THE TOPOLOGY OF RESOLUTION TOWERS

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ABSTRACT. An obstruction theory is given to determine when a space has a resolution tower. This can be used to decide whether or not the space is homeomorphic to a real algebraic set.

In [AK_1, AK_2] we have introduced topological resolution towers. These are certain topological structures on smooth stratified spaces. As the name suggests, such a structure gives a resolution of the space. Underlying topological spaces of real algebraic sets naturally have this structure [AK_3]. In fact, a certain converse of this is also true [AK_4]. These structures impose new obstructions for a space to be a real algebraic set. In this paper we outline an obstruction theory for stratified spaces to admit these structures. This theory is similar to one suggested by Sullivan [S]. As an application in [AK_5] all real algebraic sets of dimension $\leqslant 3$ are topologically classified. The main theorem of this paper is Theorem 1.2, which is a generalization of our previous spine theorem (Proposition 4.2 of [AK_1]). Theorem 1.2 says that a stratified space X with this structure bounds if and only if the structure on X extends to cone(X). This theorem makes the obstruction theory of this paper work.

Definitions. In this paper a manifold means a smooth manifold and a *closed manifold* means a compact smooth manifold without boundary. A proper codimension one submanifold S of a closed manifold V is called a *separating submanifold* if S separates V. See Figure 1. If A is a compact codimension zero submanifold of a compact manifold we define (see Figure 2)

$$\partial_-(A) = \partial(A) \cap \partial V,$$
$$\partial_+(A) = \partial(A) - \operatorname{int}\partial_-(A).$$
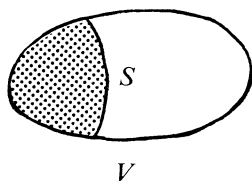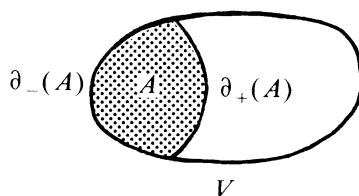
FIGURE 1

FIGURE 2
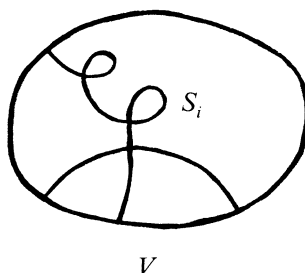
FIGURE 3

DEFINITION. Let V be a compact manifold; then a *tico* $\mathfrak{A}$ in V is a collection of properly immersed codimension one compact submanifolds of V in general position meeting ∂V transversally. Here proper means if $S \in \mathfrak{A}$ and $\partial S \neq \varnothing$, then $\partial S = \partial V \cap S$. If $\mathfrak{A} = \{ S_i \}_{i=1}^n$, each S_i is called *a sheet* of $\mathfrak{A}$. We say $\mathfrak{A}$ is a *regular tico* if each S_i is a properly imbedded submanifold. See Figure 3. We define $\partial \mathfrak{A} = \{ \partial S_i \}_{i=1}^n$; hence $\partial \mathfrak{A}$ is a tico in ∂V. Given ticos $\mathfrak{A} = \{ S_i \}$, $\mathfrak{B} = \{ R_j \}$ in general position, define $\mathfrak{A} \cap \mathfrak{B} = \{ S_i \cap R_j \}$. If R is a smooth manifold which is in general position with all S_i, define $\mathfrak{A} \cap R = \{ S_i \cap R \}$. If L is a manifold define $\mathfrak{A} \times L = \{ S_i \times L \}$, $\mathfrak{A} \times L$ is a tico in $V \times L$. We define the realization of a tico $\mathfrak{A} = \{ S_i \}_{i=1}^n$ to be $|\mathfrak{A}| = \bigcup_{i=1}^n S_i$ (notice $\{ |\mathfrak{A}| \}$ itself is a tico consisting of a single sheet).

A *closed tico* is a tico $\mathfrak{A}$ such that $\partial S = \varnothing$ for all $S \in \mathfrak{A}$. In particular, all ticos in a closed manifold are closed ticos. A *separating tico* is a regular tico such that each $S \in \mathfrak{A}$ is a separating submanifold of V. In this paper we use the symbol $\approx$ for stratified space isomorphism, so in particular if the stratified spaces are smooth manifolds, then $\approx$ denotes diffeomorphism. If V is a compact manifold with boundary, we call a tico $\mathfrak{A}$ in V a *spine* of V if

(a) $\partial \mathfrak{A} = \varnothing$,
(b) V is a regular neighborhood of $|\mathfrak{A}|$.

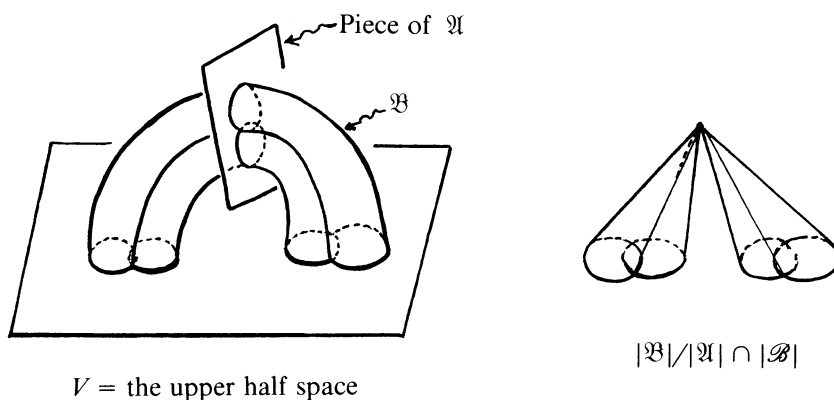$|\mathfrak{B}|/|\mathfrak{A}| \cap |\mathfrak{B}|$

$V =$ the upper half space

FIGURE 4

(b) implies that $V - |\mathfrak{A}| \approx \partial V \times (0,1]$, that is, $V/|\mathfrak{A}| \approx \text{cone}(\partial V)$. If $\mathfrak{A} \cup \mathfrak{B}$ is a tico in V with $\partial \mathfrak{A} = \varnothing$ and $\partial \mathfrak{B} \neq \varnothing$, we say $\mathfrak{A}$ is a *spine* of $\mathfrak{B}$ if $|\mathfrak{B}|/|\mathfrak{A}| \cap |\mathfrak{B}| \approx \text{cone}|\partial \mathfrak{B}|$. See Figure 4.

If $\mathfrak{A} = \{ S_i \}_{i=1}^n$ is a tico then $|\mathfrak{A}|$ is a stratified space $\bigcup M_j$ in a natural way, where $M_j = \{ x \in |\mathfrak{A}| \, | \, \alpha(x) = j \}$, and $\alpha(x) = \sum_{i=1}^n \alpha_i(x)$. $\alpha_i(x)$ is the multiplicity of S_i at x. (Thus $\alpha_i(x) = 0$ if $x \notin S_i$, $\alpha_i(x) = 1$ if S_i is imbedded near x, $\alpha_i(x) = 2$ if S_i has a double point at x, etc). In particular, if $\mathfrak{A}$ is regular then $\alpha(x)$ is the number of sheets S_i passing through x and M_j is the set of points which are contained in exactly j sheets. We also have a stratification on V, namely $V = (V - |\mathfrak{A}|) \cup \bigcup M_j$ which we will refer to as the stratification of $(V, \mathfrak{A})$.

In this paper sometimes we will have to consider a generalization of regular ticos, so we make the following definition.
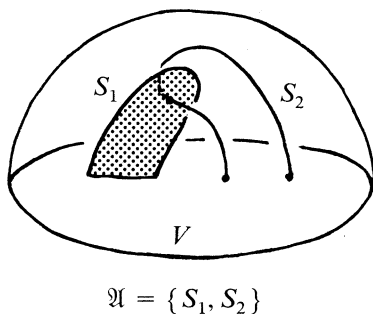

$$\mathfrak{A} = \{ S_1, S_2 \}$$

FIGURE 5

DEFINITION. Let V be a compact manifold. A *tim* $\mathfrak{A}$ in V is a collection of transversally intersecting imbedded compact submanifolds of V, such that each $S \in \mathfrak{A}$ meets ∂V transversally along a compact codimension zero submanifold of ∂S. We call the elements of $\mathfrak{A}$ sheets of $\mathfrak{A}$. Define $\mathfrak{A}|_{\partial V} = \{ \partial_- S \}_{S \in \mathfrak{A}}$, $\mathfrak{A}|_{\partial V}$ is a tim in ∂V. Define $|\mathfrak{A}| = \bigcup_{S \in \mathfrak{A}} S$, and call it the *realization of tim* $\mathfrak{A}$. Figure 5 gives a picture of a two-sheeted tim in the upper half ball.

Now we have to talk about several notions of transversality. Let $f: V \to W$ be a smooth map between smooth manifolds and let $\mathfrak{B} = \{ S_i \}$ be a tim in W; then we say f is *transverse* to $\mathfrak{B}$ if f is transverse to all possible intersections of sheets of $\mathfrak{B}$. In this case we can pull back the tim $\mathfrak{B}$ to the tim $f^*\mathfrak{B} = \{ f^{-1}(S_i) \}$ in V. Now in addition assume that $\mathfrak{A}$ is a tico in V; then we say $f: V \to W$ is *tico-transverse* to $\mathfrak{B}$ if the restriction of f to every stratum of V induced by $\mathfrak{A}$, is transverse to $\mathfrak{B}$. In this case the pull-back tim $f^*\mathfrak{B}$ is transverse to $\mathfrak{A}$ in V. Similarly, we call a smooth map $f: |\mathfrak{A}| \to W$ *tico-transverse* to $\mathfrak{B}$ if the restriction of f to every stratum of $|\mathfrak{A}|$ is transverse to $\mathfrak{B}$.

DEFINITION. Let $\mathfrak{A}$ be a tico in a smooth manifold V. A *subtim* of $\mathfrak{A}$ is a collection $\mathfrak{C}$ of subsets of $|\mathfrak{A}|$ such that $\mathfrak{C} = \{ B \cap |\mathfrak{A}| \, | \, B \in \mathfrak{B} \}$ for some tim $\mathfrak{B}$ in
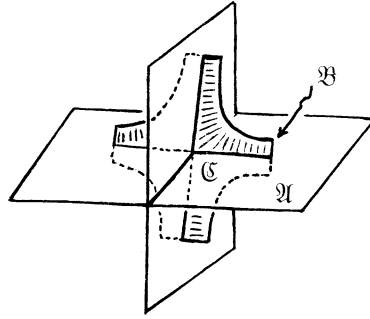
FIGURE 6

general position with $\mathfrak{A}$. See Figure 6. Note that a subtim is not itself a tim. The tim $\mathfrak{B}$ will be called a *normal thickening* of $\mathfrak{C}$. Note that normal thickenings are far from unique. The following lemma gives a useful criterion for when a subset of $|\mathfrak{A}|$ is a subtim. Let $g: N \to V$ be the immersion associated to $\mathfrak{A}$. That is, $g(N) = |\mathfrak{A}|$ and g is one-to-one on a dense subset of N. In [$\mathbf{AK}_2$], we defined a pull-back tico $g^*(\mathfrak{A})$ in N.

LEMMA 1.1 (NORMAL THICKENING LEMMA). *A finite collection $\mathfrak{C}$ of subsets of $|\mathfrak{A}|$ is a subtim of $\mathfrak{A}$ if and only if $\mathfrak{D} = \{ g^{-1}(C) \,|\, C \in \mathfrak{C} \}$ is a tim in N in general position with $g^*(\mathfrak{A})$. In case $\mathfrak{A}$ is regular this is equivalent to saying that for each sheet S of $\mathfrak{A}$, $\{ S \cap C \,|\, C \in \mathfrak{C} \}$ is a tim in S which is in general position with the tico $S \cap \mathfrak{A} = \{ S \cap A \,|\, A \in \mathfrak{A} - \{S\} \}$.*

PROOF. If $\mathfrak{C} = \{ |\mathfrak{A}| \cap B \,|\, B \in \mathfrak{B} \}$ for some tim $\mathfrak{B}$ in general position with $\mathfrak{A}$, then it is easy to check that $\mathfrak{D} = \{ g^{-1}(B) \,|\, B \in \mathfrak{B} \}$ is a tim in N in general position with $g^*(\mathfrak{A})$. To prove the converse, suppose we know $\mathfrak{D}$ is a tim in N transverse to $g^*(\mathfrak{A})$. By induction we may assume there is a tim $\mathfrak{B}'$ in general position with $\mathfrak{A}$ and a $C \in \mathfrak{C}$ so that $\mathfrak{C} \cup \{ \partial_+ C \} - \{ C \} = \{ |\mathfrak{A}| \cap B \,|\, B \in \mathfrak{B}' \}$, where $\partial_+ C = g(\partial_+ (g^{-1}(C)))$. Now for every $z \in C$ it is not hard to find a chart $\phi_z : (\mathbf{R}^m, 0) \to (V, z)$ (or $\phi_z : (\{ x \in \mathbf{R}^m \,|\, x_m \geqslant 0 \}, 0) \to (V, z)$ if $z \in \partial V$) so that ϕ_z^{-1} of everything in sight is linear. That is, $\phi_z^{-1}(|\mathfrak{A}|) = \bigcup_{i=1}^a \mathbf{R}_i^m$ (where $\mathbf{R}_i^m = \{ x \in \mathbf{R}^m \,|\, x_i = 0 \}$) and for each $B \in \mathfrak{B}'$ either $z \notin B$ or $\phi_z^{-1}(B) = \bigcap_{i=b}^c \mathbf{R}_i^m \cap$ (domain of ϕ_z) (if $z \in B - \partial_+ B$) or $\phi_z^{-1}(B) = \bigcap_{i=b}^{c-1} \mathbf{R}_i^m \cap \{ x \in \mathbf{R}^m \,|\, x_c \geqslant 0 \} \cap$ (domain of ϕ_z) (if $z \in \partial_+ B$) where b and c depend on B. Furthermore, there is some linear subspace $S_z \subset \mathbf{R}^m$ transverse to $\phi_z^{-1}(|\mathfrak{A}|)$ and a smooth $\theta_z : S_z \cap \phi_z^{-1}(|\mathfrak{A}|) \to S_z^{\perp}$ so that

$$\phi_z^{-1}(C) = \left\{ x + \theta_z(x) \,|\, x \in S_z \cap \phi_z^{-1}(|\mathfrak{A}|) \right\}.$$

So by covering C with a finite number of such charts and by extending each θ_z to a neighborhood S_z' of $S_z \cap \phi_z^{-1}(|\mathfrak{A}|)$ in $S_z \cap$ domain ϕ_z we may find some manifold $B' \subset V$ so that $\phi_z^{-1}(B') = \{ x + \theta_z(x) \,|\, x \in S_z' \}$. Then $\mathfrak{B} = \mathfrak{B}' \cup \{ B' \} - \{ \partial_+ B' \}$ is a tim in general position with $\mathfrak{A}$ and $\mathfrak{C} = \{ |\mathfrak{A}| \cap B \,|\, B \in \mathfrak{B} \}$. $\square$
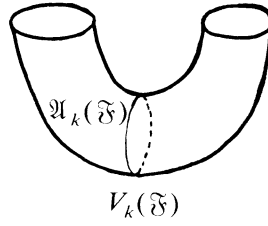
FIGURE 7

DEFINITION. Let $(V, \mathfrak{A})$, $(W, \mathfrak{B})$ be two closed smooth manifolds with ticos. A smooth map $f: V \to W$ is called a *tico map* if the following is true. for any $q \in V$ and $f(q) \in W$ pick charts $\psi: (\mathbf{R}^m, 0) \to (V, q)$, $\theta: (\mathbf{R}^n, 0) \to (W, f(q))$ so that $\psi^{-1}(|\mathfrak{A}|) = \bigcup_{i=1}^{a} \mathbf{R}_i^m$, $\theta^{-1}(|\mathfrak{B}|) = \bigcup_{i=1}^{b} \mathbf{R}_i^n$ and $f\psi(\mathbf{R}^m) \subset \theta(\mathbf{R}^n)$. Let $k_i(x)$ be the ith coordinate of $\theta^{-1} f\psi(x)$. Then f is a tico map if there are nonnegative integers α_{ij}, $1 \leqslant i \leqslant b$, $1 \leqslant j \leqslant a$ and smooth functions $\phi_i: \mathbf{R}^m \to \mathbf{R}$ such that $k_i(x) = \prod_{j=1}^{a} x_j^{\alpha_{ij}} \phi_i(x)$ for all x near 0 and $\phi_i(0) \neq 0$ for $i = 1, 2, \ldots, b$. Let $\mathfrak{C} \subset \mathfrak{A}$; then we say a map $g: |\mathfrak{C}| \to W$ is a *mico* if there is a neighborhood U of $|\mathfrak{C}|$ in V and a tico map $f: (U, U \cap \mathfrak{A}) \to (W, \mathfrak{B})$ such that $g = f|$. Similarly we can define ticos and micos in case V and W are not closed by allowing coordinate charts from upper half Euclidean spaces.

We are now ready to define resolution towers, which is the basic object of this paper. A *resolution tower* is a collection $\mathfrak{F} = \{V_i, \mathfrak{A}_i, p_i\}_{i=0}^{n}$ where $(V_i, \mathfrak{A}_i)$, $i = 0, \ldots, n$, are compact smooth manifolds with ticos and $p_i = \{p_{ji}\}_{j=0}^{i-1}$ are collections of smooth maps $p_{ji}: V_{ji} \to V_j$ so that $V_{ji} = |\mathfrak{A}_{ji}|$ for some $\mathfrak{A}_{ji} \subset \mathfrak{A}_i$ and

(1) $p_{ji}(V_{ji} \cap V_{ki}) \subset V_{kj}$ for $k < j < i$,

(2) $p_{kj} \circ p_{ji}|_{V_{ji} \cap V_{ki}} = p_{ki}|_{V_{ji} \cap V_{ki}}$,

(3) $p_{ji}^{-1}(\bigcup_{k<m} V_{kj}) = V_{ji} \cap (\bigcup_{k<m} V_{ki})$,

(4) $\mathfrak{A}_i = \bigcup_{j<i} \mathfrak{A}_{ji}$, and $\mathfrak{A}_{ji} \cap \mathfrak{A}_{ki} = \emptyset$ when $j \neq k$,

(5) $p_{ji}^{-1}(\partial V_j) = \partial V_i \cap V_{ji}$.

A resolution tower is a purely topologically defined object; to emphasize this, we will sometimes call it a *topological resolution tower*. We will denote $V_i = V_i(\mathfrak{F})$, $\mathfrak{A}_i = \mathfrak{A}_i(\mathfrak{F})$, $\mathfrak{A}_{ji} = \mathfrak{A}_{ji}(\mathfrak{F})$, $V_{ji} = V_{ji}(\mathfrak{F})$, and $p_{ji} = p_{ji}(\mathfrak{F})$. We define $n = $ height$(\mathfrak{F})$. We call a topological resolution tower $\mathfrak{F}$ *seminormal* (*normal*) if whenever W is any component of $V_i(\mathfrak{F})$ with $\partial W \neq \emptyset$ and U is a component of $W - |\mathfrak{A}_i(\mathfrak{F})|$, then $U \cap \partial W \neq \emptyset$ (U meets all the components of ∂W). Clearly normal implies seminormal but not the converse; for example, Figure 7 satisfies the seminormal condition but not the normality condition. In case all $\partial V_i = \emptyset$ seminormality and normality are vacuous conditions.

We can talk about some additional properties of resolution towers (see [AK$_2$]); the ones we use in this paper are:

R—Each tico $\mathfrak{A}_i$ is regular.

M—Each p_{ji} is a mico.

U—Each $p_{ji}|_S$ is a smooth submersion for every stratum S of $(V_i, \mathfrak{A}_i)$ with $S \subset V_{ji} - \bigcup_{k<j} V_{ki}$.

S—Each p_{ji} is a submersive mico, in particular $p_{ji}|_S$ is a smooth submersion for every stratum S of $(V_i, \mathfrak{A}_i)$ (see [$\mathbf{AK_2}$]).

F—Each $(V_i, \mathfrak{A}_i)$ is full. This means that if S is V_i or any intersection of sheets of $\mathfrak{A}_i$, $H_*(S; \mathbb{Z}/2\mathbb{Z})$ is generated by imbedded submanifolds of S.

For example, we will say $\mathfrak{F}$ is type R it it satisfies R. Sometimes we will combine these properties; for instance, a resolution tower $\mathfrak{F}$ of type RM satisfies both the properties R and M.

If $\mathfrak{F} = \{V_i, \mathfrak{A}_i, p_i\}$ then we define $\partial \mathfrak{F} = \{\partial V_i, \partial \mathfrak{A}_i, p_i|\}$. We call $\mathfrak{F}$ closed if $\partial \mathscr{F} = \varnothing$. For $0 \leqslant k \leqslant n$ we denote $\mathfrak{F}_k = \{V_i, \mathfrak{A}_i, p_i\}_{i=0}^k$.

An *algebraic resolution tower* is a resolution tower such that each V_i is a nonsingular algebraic set and each sheet of $\mathfrak{A}_i$ is a nonsingular algebraic subset and each p_{ji} is an entire rational function. We define $\mathscr{T}$ to be the set of all topological resolution towers and $\mathscr{A}$ to be the set of all algebraic resolution towers. When we put subscripts R, M, U, S, F to $\mathscr{T}$ or $\mathscr{A}$, they will denote the subsets of $\mathscr{T}$ or $\mathscr{A}$ of that type. For example, $\mathscr{T}_{RM}$ is the set of resolution towers of type RM. Clearly, we can identify $\mathscr{A} \hookrightarrow \mathscr{T}$ via the forgetful map. Note that $\mathscr{A}_R = \mathscr{A}$ and $\mathscr{T}_S \subset \mathscr{T}_{MU}$, $\mathscr{A}_S \subset \mathscr{A}_{MU}$.

If $\mathfrak{F} \in \mathscr{T}$ we define the *realization of* $\mathfrak{F}$ to be

$$|\mathfrak{F}| = \bigcup_{i=0}^{n} V_i / x \sim p_{ji}(x).$$

That is, $|\mathfrak{F}|$ is obtained by identifying the points $x \sim p_{ji}(x)$ for all $x \in V_{ji}$ in the disjoint union $\bigcup_{i=0}^n V_i$. If $\mathfrak{F} \in \mathscr{T}$ then $|\mathfrak{F}|$ is a stratified space with strata $\{V_j - \bigcup_{r<j} V_{rj}\}$. We define $\dim(\mathfrak{F})$ to be the dimension of $|\mathfrak{F}|$, i.e. the dimension of the highest dimensional stratum.

If $\mathfrak{F} \in \mathscr{A}_U$ then $|\mathfrak{F}|$ is P.L. homeomorphic to a real algebraic set whose singular stratification coincides with the smooth stratification of $|\mathfrak{F}|$ [$\mathbf{AK_4}$]. (We say that two stratified sets are P.L. homeomorphic if they have smooth subdivisions which are isomorphic as stratified sets.) Note that an algebraic set has a canonical stratification up to P.L. homeomorphism since any two semialgebraic stratifications have a common subdivision. We know that any algebraic set is P.L. homeomorphic to $|\mathfrak{F}|$ for some $\mathfrak{F} \in \mathscr{A}$; in fact $\mathfrak{F} \in \mathscr{A}_{MUF}$ [$\mathbf{AK_3}$].

We adopt the convention that if $\mathscr{S}$ is any subset of $\mathscr{T}$ or $\mathscr{A}$, then

$$|\mathscr{S}| = \{|\mathfrak{F}| \mid \mathfrak{F} \in \mathscr{S}, \partial \mathfrak{F} = \varnothing\}.$$

So up to P.L. homeomorphism, we have

$$\{\text{Real algebraic sets}\} = |\mathscr{A}_{MUF}|.$$

Let $\tau : |\mathscr{A}| \hookrightarrow |\mathscr{T}|$ be the forgetful map. In $[\mathbf{AK_2}]$ it is shown that for any $\mathfrak{F} \in \mathscr{T}$ there is a regular $\mathfrak{F}' \in \mathscr{T}$ with $|\mathfrak{F}'| = |\mathfrak{F}|$ and $\mathfrak{F}'$ has the same type as $\mathfrak{F}$. In $[\mathbf{AK_4}]$ it is shown that τ maps $|\mathscr{A}_{SF}|$ onto $|\mathscr{T}_{RSF}|$. So up to P.L. homeomorphism, we have the following commutative diagram:

where the horizontal maps are restrictions of τ, and j is induced by composition. Hence proving j (or less likely τ') onto would give a complete topological characterization of real algebraic sets. Anyhow, the elements of $|\mathscr{T}_{RS}|$ give a good picture of what real algebraic sets look like. In this paper we give an obstruction theory to decide which stratified spaces lie in $|\mathscr{T}_{RS}|$. Many of the results of this paper remain valid when we replace $\mathscr{T}_{RS}$ with some other subsets of $\mathscr{T}$. For clarity we will stick with $\mathscr{T}_{RS}$ throughout the paper, and leave it to the reader to verify these possible variations of the results. For brevity we will denote $TR = \mathscr{T}_{RS}$ and $AR = \mathscr{A}_S$.

We are now ready to state our main result.

THEOREM 1.2. *If* $\mathfrak{G} \in TR$ *and* $\mathfrak{G} = \partial \mathfrak{F}'$ *for some seminormal* $\mathfrak{F}' \in TR$, *then there exists* $\mathfrak{F} \in TR$ *such that*

(i) $\partial \mathfrak{F} = \mathfrak{G}$,

(ii) $|\mathfrak{F}| \approx \text{cone}|\mathfrak{G}|$.

Above $\approx$ means a stratified space isomorphism. This theorem is the obvious generalization of the classical spine theorem which allowed us to classify the real algebraic sets with isolated singularities $[\mathbf{AK_6}]$. Notice in particular (ii) implies that $\mathfrak{F}$ must be seminormal.

Theorem 1.2 is the basic result that allows us to define an obstruction theory of putting TR structures on stratified spaces. Let Y be a stratified space of dimension n such that each connected smooth stratum has a trivial normal bundle. Let Z be the s-skeleton of Y, i.e. Z is the union of i-strata for all $i \leqslant s$. Suppose $Y - Z$ has a TR structure respecting the trivializations, so there is a $\mathfrak{F} \in TR$ with $|\mathfrak{F}| = Y - Z$. Also by assumption if σ is an open connected s-stratum then $\mathfrak{F}$ restricts to $\mathfrak{F}' \times \sigma$

FIGURE 8   FIGURE 9

on the boundary of the tubular neighborhood of σ for some $\mathfrak{F}' \in TR$, in particular $|\mathfrak{F}' \times \sigma| = |\mathfrak{F}'| \times \sigma$. Let $Z' = Z - \sigma$. If we can find $\mathfrak{F}'' \in TR$ with $\partial \mathfrak{F}'' = \mathfrak{F}'$ and $|\mathfrak{F}''| = $ cone on $|\mathfrak{F}'|$, then we can extend $\mathfrak{F}$ to $Y - Z'$ across σ by $\mathfrak{F} \cup_\partial (\mathfrak{F}'' \times \sigma)$. See Figure 8.

Theorem 1.2 reduces the awkward question of finding such an $\mathfrak{F}''$ to a bordism problem, namely it suffices to prove that $\mathfrak{F}'$ bounds a seminormal resolution tower. In case Z is a point, then the theorem roughly says that $|\mathfrak{F}'|$ can occur as a link of a point in an algebraic variety if $\mathfrak{F}'$ bounds.

As an example, let Y be the stratified set in Figure 9. Let Z_i be the ith skeleton of Y. Then $Y - Z_1$ has a TR structure since it is smooth. The link of σ_1 is four points; they bound two arcs. Let $\mathfrak{F}_1$ be the resolution tower

Then we can extend the TR structure on $Y - Z_1$ by $\mathfrak{F}_1 \times \sigma_1$ across σ_1, since

$$|\mathfrak{F}_1| = \quad \times .$$

So we have a TR structure on $Y - Z_0$ and $Z_0 = \{\sigma_0, \sigma_0'\}$. This induces TR structures on the links of σ_0 and σ_0' by

Let

where p_{12} is the folding map, p_{01} and p_{02} are collapsing maps. Then $\partial \mathfrak{F}_2' = \mathfrak{G}$ and $\mathfrak{F}_2'$ is seminormal. Theorem 1.2 tells us that there is an $\mathfrak{F}_2 \in TR$ such that $\partial \mathfrak{F}_2 = \mathfrak{G}$
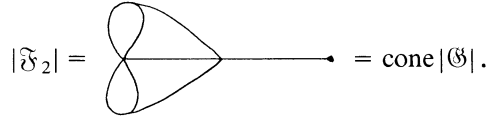
and $|\mathfrak{F}_2| \approx \text{cone}|\mathfrak{G}|$, so we can extend the TR structure across σ_0 and σ_0' to all of Y by adding two copies of $\mathfrak{F}_2$. For example, we can take
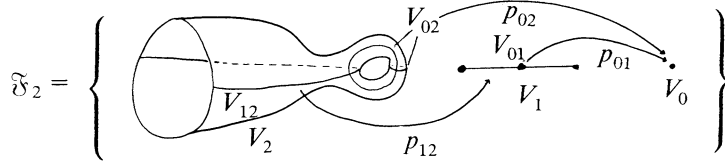
since

In higher dimensions, it is often harder to find $\mathfrak{F}_2$ than $\mathfrak{F}_2'$, which is any seminormal bordism with $\partial \mathfrak{F}_2' = \mathfrak{G}$, but Theorem 1.2 allows us to find such $\mathfrak{F}_2$ from $\mathfrak{F}_2'$.

2. Properties of resolution towers.

It is shown in [$\mathbf{AK_2}$] that if $\mathfrak{F} \in \mathscr{T}_{MU}$, then there exists $\mathfrak{F}' \in \mathscr{T}_{RMU}$ with $|\mathfrak{F}'| = |\mathfrak{F}|$. Here we study some further topological properties of resolution towers. Our main goal is to be able to modify resolution towers to nicer ones as in the above result.

PROPOSITION 2.1. *Let* $\mathfrak{F} \in TR$ *be seminormal with* $V_0(\mathfrak{F}) = a$ *point. Let* k *be the smallest integer* $\geqslant 1$ *such that* $V_k(\mathfrak{F})$ *has closed manifold components (if it exists). Then there exists a seminormal* $\mathfrak{F}' \in TR$ *such that*

(1) $\partial \mathfrak{F}' = \partial \mathfrak{F}$,

(2) $\mathfrak{F}'_{k-1} = \mathfrak{F}_{k-1}$,

(3) $V_i(\mathfrak{F}') = $ *the union of connected components of* $V_i(\mathfrak{F})$ *with nonempty boundary for* $i \geqslant k$.

(4) $|\mathfrak{A}_i(\mathfrak{F}')| = |\mathfrak{A}_i(\mathfrak{F})| \cap V_i(\mathfrak{F}')$, *for* $i \geqslant k$.

PROOF. Let $\mathfrak{F} = \{V_i, \mathfrak{A}_i, p_i\}_{i=0}^n$. Define $\mathfrak{F}'_{k-1} = \mathfrak{F}_{k-1}$ and for each $i \geqslant k$ let V_i' be the union of connected components of V_i which have nonempty boundary. For all $1 \leqslant j < i$ let $V_{ji}' = p_{ji}^{-1}(V_j') \cap V_i'$, and $p_{ji}' : V_{ji}' \to V_j'$ be the restriction of p_{ji}. Let $V_{0i}' = V_i' \cap \bigcup_{j=1}^{i-1}(V_{ji} - V_{ji}')$, and $p_{0i}' : V_{0i}' \to V_0'$ be the constant map. Let $\mathfrak{A}_{ji}'$ be the sheets of V_{ji}' and $\mathfrak{A}_i' = \bigcup_{j=0}^{i-1} \mathfrak{A}_{ji}'$ and $p_i' = \{p_{0i}', \ldots, p_{i-1,i}'\}$. Then $\mathfrak{F}' = \{V_i', \mathfrak{A}_i', p_i'\}_{i=0}^n$ gives the required resolution tower. $\square$

DEFINITION. Let $\mathfrak{F}, \mathfrak{F}' \in TR$ with $\text{height}(\mathfrak{F}') \leqslant \text{height}(\mathfrak{F})$. Then a transverse tower map $\phi : \mathfrak{F}' \to \mathfrak{F}$ is a collection of maps $\phi_i : V_i(\mathfrak{F}') \to V_i(\mathfrak{F})$ for $i = 0, 1, \ldots, \text{height}(\mathfrak{F}')$ which are tico-transverse to $\mathfrak{A}_i(\mathfrak{F})$ such that for all $0 \leqslant j < i \leqslant \text{height}(\mathfrak{F}')$

(a) $\mathfrak{A}_{ji}(\mathfrak{F}') = \phi_i^* \mathfrak{A}_{ji}(\mathfrak{F})$,

(b) The following commutes:

$$
\begin{array}{ccc}
V_{ji}(\mathfrak{F}') & \xrightarrow{\phi_i} & V_{ji}(\mathfrak{F}) \\
\downarrow p_{ji}(\mathfrak{F}') & & \downarrow p_{ji}(\mathfrak{F}) \\
V_j(\mathfrak{F}') & \xrightarrow{\phi_j} & V_j(\mathfrak{F})
\end{array}
$$

where $V_{ji}(\mathfrak{F}') = |\mathfrak{A}_{ji}(\mathfrak{F}')|$.

We say $\mathfrak{F}'$ is a *subtower* of $\mathfrak{F}$ if all ϕ_i are inclusions. We call a subtower $\mathfrak{F}'$ a *codimension k subtower* if all $V_i(\mathfrak{F}')$ are codimension k submanifolds of $V_i(\mathfrak{F})$.

DEFINITION. Let $\mathfrak{F} \in TR$ with $n = \text{height}(\mathfrak{F})$. Then $\mathscr{A} = \{A_j\}_{j=k}^{m}$ where $k \leqslant m \leqslant n$ is called a *compatible family of submanifolds* of $\mathfrak{F}$ if each A_j is a compact smooth submanifold of $V_j(\mathfrak{F})$ (it can be empty) such that

(1) A_j is transverse to $\mathfrak{A}_j(\mathfrak{F})$, for all $j \geqslant k$.

(2) $p_{ji}(\mathfrak{F})$ is tico-transverse to A_j, for all $i > j \geqslant k$.

(3) $p_{ji}(\mathfrak{F})^{-1}(A_j) = V_{ji}(\mathfrak{F}) \cap A_i$, $p_{ji}(\mathfrak{F})^{-1}(\partial A_j) = V_{ji}(\mathfrak{F}) \cap \partial A_i$, for $k \leqslant j < i \leqslant m$. More generally we define $\mathscr{A} = \{\mathscr{B}_i\}_{i=k}^{m}$ to be a *compatible family of tims* in $\mathfrak{F}$ if each $\mathscr{B}_i = \{A_{ji}\}_{j \in \Lambda}$ is a tim in $V_i(\mathfrak{F})$ such that each $\mathscr{A}_\alpha = \{A_{\alpha i}\}_{i=k}^{m}$ is a compatible family of submanifolds of $\mathfrak{F}$ for all $\alpha \subset \Lambda$ where $A_{\alpha i} = \bigcap_{j \in \alpha} A_{ji}$. In particular, *compatible family of ticos* is defined. Also we define $\mathscr{A}|_{\partial \mathfrak{F}}$ to be the compatible family of tims $\{\mathscr{B}_i|_{\partial \mathfrak{F}}\}$.

LEMMA 2.2. *Let $\mathfrak{F} \in TR$, $m \leqslant \text{height}(\mathfrak{F})$, and let $\mathscr{A} = \{A_i\}_{i=k}^{m-1}$ be a compatible family of submanifolds in $\mathfrak{F}$. Then there exists a smooth submanifold A_m in $V_m(\mathfrak{F})$ such that $\mathscr{A}' = \{A_i\}_{i=k}^{m}$ is a compatible family of submanifolds in $\mathfrak{F}$.*

PROOF. Let $\mathscr{B}_{jm} = p_{jm}(\mathfrak{F})^*\{A_j\}$ and $\mathscr{B}_m = \bigcup_{j=k}^{m-1} \mathscr{B}_{jm}$. Then we claim that $\mathscr{B}_m$ is a subtim in $\bigcup_{j \geqslant k} \mathfrak{A}_{jm}(\mathfrak{F})$. Given this, we can take a normal thickening A_m of $\mathscr{B}_m$ in $V_k(\mathfrak{F})$. So by construction $\mathscr{A}'$ satisfies the conditions (1) and (3) of a compatible family. To check (2) take any connected component S_0 of a stratum of V_i with $S_0 \subset V_{mi}$. We must show that $p_{mi}|: S_0 \to V_m$ is transverse to A_m. Let S_1 be the stratum of V_m with $S_1 \supset p_{mi}(S_0)$. Suppose first that $S_0 \subset V_{ji}$ for some $j < m$. Then $S_1 \subset |\mathfrak{A}_m(\mathfrak{F})|$ so since $p_{ji}|: S_0 \to V_j$ is transverse to A_j, $p_{ji}|_{S_0} = p_{jm} \circ p_{mi}|_{S_0}$ and $S_1 \cap A_m = S_1 \cap p_{jm}^{-1}(A_j)$ we know that $p_{mi}|_{S_0}$ must be transverse to A_m. Now suppose that $S_0 \subset V_{mi} - \bigcup_{j < m} V_{ji}$. Because $\mathfrak{F}$ has type U we are done since p_{mi} submerses S_0 to $V_m - |\mathfrak{A}_m|$. Now it remains to show that $\mathscr{B}_m$ is a subtim in $\bigcup_{j \geqslant k} \mathfrak{A}_{jm}(\mathfrak{F})$, but this follows from Lemma 1.1. See Figure 10. $\square$

LEMMA 2.3. *Let $\mathfrak{F} \in TR$, $m \leqslant \text{height}(\mathfrak{F})$, and let $\mathscr{A} = \{\mathscr{A}_i\}_{i=k}^{m-1}$ and $\mathscr{B} = \{\mathscr{B}_i\}_{i=k}^{m}$ be compatible families of tims in $\mathfrak{F}$ with $\mathscr{B}_i \subset \mathscr{A}_i$ for all $i < m$. Then there exists a tim $\mathscr{A}_m$ in $V_m(\mathfrak{F})$ such that $\mathscr{B}_m \subset \mathscr{A}_m$ and $\mathscr{A}' = \{\mathscr{A}_i\}_{i=k}^{m}$ is a compatible family of tims. Furthermore, if there is a tim $\mathscr{C}_m$ in $\partial V_m(\mathfrak{F})$ such that $\mathscr{A}|_{\partial \mathfrak{F}} \cup \{\mathscr{C}_m\}$ is a compatible family of tims in $\partial \mathfrak{F}$, then we can choose $\mathscr{A}_m$ such that $\mathscr{A}_m|_{\partial \mathfrak{F}} = \mathscr{C}_m$.*
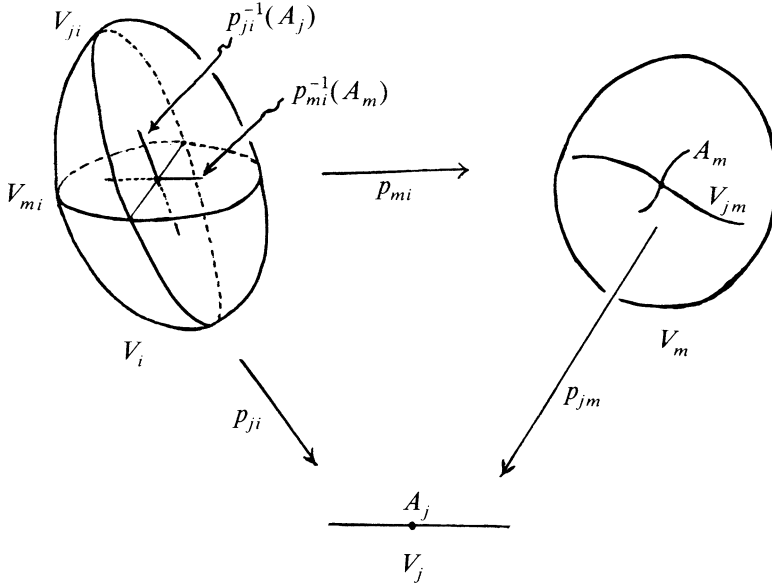
FIGURE 10

PROOF. The proof follows by applying Lemma 2.2 to the sheets of $\mathscr{A}$. To get the relative version, recall that the sheets of $\mathscr{A}_m$ are obtained by taking normal thickenings and there is relative existence for normal thickenings. □

LEMMA 2.4. *Let* $\mathfrak{F} \in TR$, $m \leqslant$ height($\mathfrak{F}$). *Let* $\mathscr{A} = \{\mathscr{A}_i\}_{i=k}^{m-1}$ *be a compatible family of tims in* $\mathfrak{F}$ *and let* $\mathscr{B} = \{\mathscr{B}_i\}_{i=k}^m$ *be a compatible family of ticos in* $\mathfrak{F}$. *Suppose* $\mathscr{B}_i \subset \mathscr{A}_i$ *for* $k \leqslant i \leqslant m - 1$ *and each sheet of each* $\mathscr{A}_i - \mathscr{B}_i$ *has codimension zero. Then there exists a tim* $\mathscr{A}_m$ *in* $V_m(\mathfrak{F})$ *so that* $\mathscr{B}_m \subset \mathscr{A}_m$, *each sheet of* $\mathscr{A}_m - \mathscr{B}_m$ *has codimension zero,* $\mathscr{A}' = \{\mathscr{A}_i\}_{i=k}^m$ *is a compatible family of tims and* $\mathscr{B}' = \{\mathscr{B}_i'\}_{i=k}^m$ *is a compatible family of ticos where* $\mathscr{B}_i' = \mathscr{B}_i \cup \{\partial_+ A \mid A \in \mathscr{A}_i - \mathscr{B}_i\}$.

PROOF. Apply Lemma 2.3 to obtain $\mathscr{A}_m$. The compatibility of the family $\{\mathscr{B}_i'\}_{i=k}^m$ then follows easily from compatibility of $\{\mathscr{A}_i\}_{i=k}^m$ and $\{\mathscr{B}_i\}_{i=k}^m$. □

Let $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}'' \in TR$, dim($\mathfrak{F}'$) = dim($\mathfrak{F}''$), such that $\partial\mathfrak{F}' = \mathfrak{G} \cup_\partial \mathfrak{G}'$ and $\partial\mathfrak{F}'' = \mathfrak{G} \cup_\partial \mathfrak{G}''$, where $\mathfrak{G}$, $\mathfrak{G}'$ and $\mathfrak{G}$, $\mathfrak{G}''$ are codimension zero subtowers of $\partial\mathfrak{F}'$ and $\partial\mathfrak{F}''$, respectively, with $\partial\mathfrak{G} = \partial\mathfrak{G}' = \partial\mathfrak{G}''$, and $\mathfrak{F} = \mathfrak{F}' \cup \mathfrak{F}''$ obtained by gluing along $\mathfrak{G}$. See Figure 11.
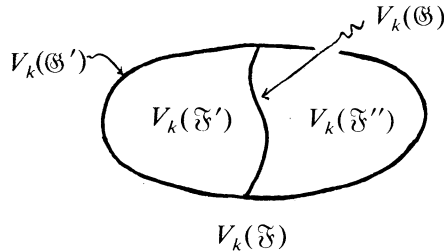

FIGURE 11

Now given these notations we have the following proposition.

PROPOSITION 2.5. *Let A_k be a compact codimension zero submanifold of $V_k(\mathfrak{F}'') \cap$ $\operatorname{int} V_k(\mathfrak{F})$ such that $A_k \cap |\mathfrak{A}_k(\mathfrak{F})| = \varnothing$. Let $\mathscr{B}^* = \{\mathscr{B}_i^*\}_{i \geqslant k}$ be a compatible family of ticos in $\mathfrak{F}'$ and let $\mathscr{B}_k$ be a tico in $V_k(\mathfrak{F}') \cup A_k$ transverse to $V_k(\mathfrak{G})$ and $\mathfrak{A}_k(\mathfrak{G})$ such that $\mathscr{B}_k \cap V_k(\mathfrak{F}') = \mathscr{B}_k^* \cup \mathscr{B}_k^{**}$ and $\mathscr{B}_k^{**}$ is a separating tico. Then after a small isotopy of A_k and $\mathscr{B}_k^{**}$ we can find a compatible family of codimension zero submanifolds $\mathscr{A} = \{A_i\}_{i \geqslant k}$ in $\mathfrak{F}''$, and ticos $\mathscr{B}_i$ in $V_i(\mathfrak{F}') \cup A_i$ for all $i > k$, and $\overline{\overline{\mathfrak{F}}} \in TR$ such that*

(a) $\overline{\overline{\mathfrak{F}}}_{k-1} = \mathfrak{F}'_{k-1}$,

(b) $V_i(\overline{\overline{\mathfrak{F}}}) = V_i(\mathfrak{F}') \cup A_i$ *for* $i \geqslant k$,

(c) $\mathfrak{A}_i(\overline{\overline{\mathfrak{F}}}) = \mathfrak{A}_i(\mathfrak{F}) \cap V_i(\overline{\overline{\mathfrak{F}}})$, $p'_i(\overline{\overline{\mathfrak{F}}}) = p_i(\mathfrak{F})|_{\mathfrak{A}_i(\overline{\overline{\mathfrak{F}}})}$,

(d) $\mathscr{B} = \{\mathscr{B}_i\}_{i \geqslant k}$ *is a compatible family of ticos in* $\overline{\overline{\mathfrak{F}}}$.

(e) $\mathscr{B}_i^* \subset \mathscr{B}_i \cap V_i(\mathfrak{F}')$ *for all* $i \geqslant k$.

Furthermore, if $\mathscr{B}_k^{**} = \{\partial_+ D \mid D \in \mathscr{D}_k\}$ *for some tim* $\mathscr{D}_k$ *with* $|\mathscr{D}_k| \cap \partial V_k(\mathfrak{G}) = \varnothing$, *then we may also conclude that* $|\mathscr{B}_i| \cap \partial V_i(\mathfrak{G}) = |\mathscr{B}_i^*| \cap \partial V_i(\mathfrak{G})$ *for all* $i \geqslant k$. *See Figure* 12.

PROOF. After a small isotopy of A_k and $\mathscr{B}_k^{**}$ we can assume that all $p_{ki}(\mathfrak{F})$'s are tico-transverse to A_k and $\mathscr{B}_k^{**}$ for all $i > k$. Let $\mathscr{D}_k$ be a codimension zero tim in $V_k(\mathfrak{F}')$ so that $\{\partial_+ D \mid D \in \mathscr{D}_k\} = \mathscr{B}_k^{**}$. By Lemma 2.4 with $m = k + 1$ and $\mathscr{A}_k = \mathscr{D}_k \cup \mathscr{B}_k^*$, there is a tim $\mathscr{A}_{k+1} = \mathscr{D}_{k+1} \cup \mathscr{B}_{k+1}^*$ so that $\{\mathscr{A}_k, \mathscr{A}_{k+1}\}$ is a compatible family. In case $|\mathscr{D}_k| \cap \partial V_k(\mathfrak{G}) = \varnothing$ we first use Lemma 2.4 to lift $\mathscr{A}_k|_{\mathfrak{G}}$ to a tim $\mathscr{D}_{k+1}^* \cup \mathscr{B}_{k+1}^*|_{\mathfrak{G}}$ in $V_{k+1}(\mathfrak{G})$ so that $|\mathscr{D}_{k+1}^*| \cap \partial V_{k+1}(\mathfrak{G}) = \varnothing$, then lift $\mathscr{A}_k|_{\mathfrak{G}'}$ to a tim $\mathscr{D}_{k+1}' \cup \mathscr{B}_{k+1}^*|_{\mathfrak{G}'}$ in $V_{k+1}(\mathfrak{G}')$ so that $|\mathscr{D}_{k+1}'| \cap \partial V_{k+1}(\mathfrak{G}) = \varnothing$. Taking unions of sheets in $\mathscr{D}_{k+1}'$ and in $\mathscr{D}_{k+1}'$ we get a tim $\mathscr{D}_{k+1}''$ in $\partial V_{k+1}(\mathfrak{F}')$ so that $\mathscr{D}_{k+1}'' \cup \mathscr{B}_{k+1}^*|_{\partial \mathfrak{F}'}$ lifts $\mathscr{A}_k|_{\partial \mathfrak{F}'}$ and $|\mathscr{D}_{k+1}''| \cap \partial V_{k+1}(\mathfrak{G}) = \varnothing$. We then pick our above lifting $\mathscr{D}_{k+1}$ so that $\mathscr{D}_{k+1}|_{\partial \mathfrak{F}'} = \mathscr{D}_{k+1}''$, in particular $|\mathscr{D}_{k+1}| \cap \partial V_{k+1}(\mathfrak{G}) = \varnothing$. Continuing in this manner we obtain a compatible family of tims $\{\mathscr{A}_i\}_{i \geqslant k}$ in $\mathfrak{F}'$ where $\mathscr{A}_i = \mathscr{D}_i \cup \mathscr{B}_i^*$, $i \geqslant k$, and $\{\mathscr{D}_i\}_{i \geqslant k}$ is a compatible family of tims in $\mathfrak{F}'$ and $|\mathscr{D}_i| \cap \partial V_i(\mathfrak{G}) = \varnothing$. So we have a compatible family of ticos in $\mathfrak{F}'$, $\mathscr{B}' = \{\mathscr{B}_i'\}_{i \geqslant k}$ where $\mathscr{B}_i' = \mathscr{B}_i^* \cup \{\partial_+ D \mid D \in \mathscr{D}_i\}$. Notice $|\mathscr{B}_i'| \cap \partial V_i(\mathfrak{G}) = |\mathscr{B}_i^*| \cap \partial V_i(\mathfrak{G})$ for all $i \geqslant k$.
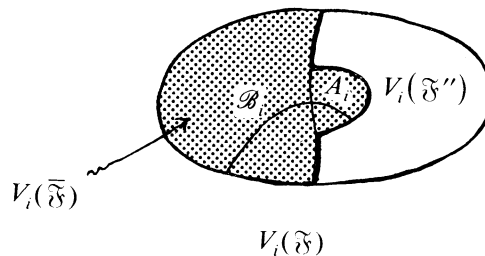

$$V_i(\mathfrak{F})$$

FIGURE 12

Form a tim $\mathscr{B}_k''$ in $\mathfrak{F}''$ with $\mathscr{B}_k'' \cap A_k = \mathscr{B}_k \cap A_k$, and $\mathscr{B}_k'' \cap V_k(\mathfrak{G}) = \mathscr{B}_k \cap V_k(\mathfrak{G})$ by taking $\mathscr{B}_k \cap (A_k \cup V_k(\mathfrak{G}))$ and adding a collar along $\mathscr{B}_k \cap (\partial_+ A_k \cup V_k(\mathfrak{G}) - \partial_-(A_k))$. By repeated applications of Lemma 2.4 we obtain a compatible family of tims $\{\mathscr{B}_i''\}_{i \geq k}$ in $\mathfrak{F}''$ with $\mathscr{B}_i'' \cap V_i(\mathfrak{G}) = \mathscr{B}_i' \cap V_i(\mathfrak{G})$. Now apply Lemma 2.2 repeatedly to A_k to obtain a compatible family of submanifolds $\{A_i\}_{i \geq k}$ in $\mathfrak{F}''$ with each A_i in general position with $\mathfrak{A}_i(\mathfrak{F}'')$ and $\mathscr{B}_i''$. Furthermore, an examination of the proof of Lemma 2.2 reveals that we may guarantee that each $A_i \cap \mathscr{B}_i''$ is a tico in A_i. (Do this by making your thickenings very thin. Actually the proof of the normal thickening lemma will do this automatically.) Then $\mathscr{A} = \{\overline{V}_i\}_{i \geq 0}$, where

$$\overline{V}_i = \begin{cases} V_i(\mathfrak{F}') & \text{for } i < k, \\ V_i(\mathfrak{F}') \cup A_i & \text{for } i \geq k, \end{cases}$$

is a compatible family of codimension zero submanifolds in $\mathfrak{F}$; hence it defines a subtower $\overline{\mathfrak{F}} = \{\overline{V}_i, \overline{\mathfrak{A}}_i, \overline{p}_i\}$ where $\overline{\mathfrak{A}}_i = \mathfrak{A}_i(\mathfrak{F}) \cap \overline{V}_i$. Then $\mathscr{B} = \{\mathscr{B}_i\}_{i \geq k}$ where $\mathscr{B}_i = |\mathscr{B}_i'| \cup_\partial |\mathscr{B}_i'' \cap A_i|$ is a compatible family of ticos in $\overline{\mathfrak{F}}$. See Figure 13.
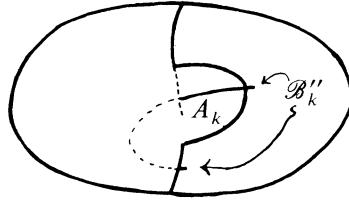
FIGURE 13

COROLLARY 2.6. *Under the same hypothesis of Proposition 2.5 and $|\mathscr{B}_i^*| \cap \partial V_i(\mathfrak{G})$ $= \varnothing$ for all $i \geq k$, $|\mathscr{D}_k| \cap \partial V_k(\mathfrak{G}) = \varnothing$ where $\mathscr{B}_k^{**} = \partial_+ \mathscr{D}_k$, and $V_0(\mathfrak{G}) = a$ point, then there exists $\overline{\mathfrak{G}} \in TR$ such that*

(a) $\partial \overline{\mathfrak{G}} = \partial \mathfrak{G}$,

(b) $\overline{\mathfrak{G}}_{k-1} = \mathfrak{G}_{k-1}$,

(c) $V_k(\overline{\mathfrak{G}}) = (V_k(\mathfrak{G}) - \text{int}(\partial_- A_k)) \cup \partial_+ A_k$,

(d) $\mathfrak{A}_{jk}(\overline{\mathfrak{G}}) = \mathfrak{A}_{jk}(\mathfrak{G})$ *for* $0 < j < k$,

(e) $\mathfrak{A}_{0k}(\overline{\mathfrak{G}}) = \mathfrak{A}_{0k}(\mathfrak{G}) \cup \partial \mathscr{B}_k$,

(f) $p_{jk}(\overline{\mathfrak{G}}) = p_{jk}(\mathfrak{G})$ *for* $0 < j < k$, $p_{0k}(\overline{\mathfrak{G}}) = $ *trivial map.*

PROOF. Let $\mathfrak{G}^* = \partial \overline{\mathfrak{F}} - \text{int}(\mathfrak{G}')$, then $\mathfrak{G}^*$ satisfies (a) and (b) and $\mathfrak{A}_i(\mathfrak{G}^*) = \mathfrak{A}_i(\mathfrak{F}) \cap V_i(\mathfrak{G}^*)$ and $p_i(\mathfrak{G}^*) = p_i(\mathfrak{F})|_{\mathfrak{A}_i(\mathfrak{G}^*)}$. Then simply define $\overline{\mathfrak{G}}$ by $\overline{\mathfrak{G}}_{k-1} = \mathfrak{G}_{k-1}^*$, $V_i(\overline{\mathfrak{G}}) = V_i(\mathfrak{G}^*)$, $\mathfrak{A}_i(\overline{\mathfrak{G}}) = \mathfrak{A}_i(\mathfrak{G}^*) \cup \partial \mathscr{B}_i$, $\mathfrak{A}_{0i}(\overline{\mathfrak{G}}) = \mathfrak{A}_{0i}(\mathfrak{G}^*) \cup \partial \mathscr{B}_i$ for all $i \geq k$, $p_{ji}(\overline{\mathfrak{G}}) = p_{ji}(\mathfrak{G}^*)$ for $0 < j < i$, and $p_{0i}(\overline{\mathfrak{G}}) = $ trivial map. Notice since A_k is disjoint from $\mathfrak{A}_k(\mathfrak{F})$, $\mathfrak{A}_k(\mathfrak{G}^*) = \mathfrak{A}_k(\mathfrak{G})$ so (c), (d), (e) are satisfied. Clearly (b) is satisfied and (a) is satisfied since $|\partial \mathscr{B}_i| \cap \partial V_i(\overline{\mathfrak{G}}) = |\mathscr{B}_i| \cap \partial V_i(\mathfrak{G}) = \varnothing$ by Proposition 2.5. $\square$

Let T^m be the m-torus; there is a basic tico in T^m which we will denote by $\mathcal{T}^m = \{T_i\}_{i=1}^m$ with each $T_i = S^1 \times \cdots \times S^1 \times 0 \times S^1 \times \cdots \times S^1$ where 0 lies in the ith slot. Let V^m be a smooth m-manifold; then $\mathcal{T}^m$ induces a tico in $V^m \# T^m$ which we will continue to denote by $\mathcal{T}^m$.

PROPOSITION 2.7. *Let $\mathfrak{G} \in TR$ be seminormal with $V_0(\mathfrak{G}) = a$ point. Let k be any integer $\leqslant$ height($\mathfrak{G}$) such that all components of $V_i(\mathfrak{G})$ have nonempty boundaries for $0 < i \leqslant k$. Then there exists $\mathfrak{G}' \in TR$ such that*

(a) $\partial \mathfrak{G}' = \partial \mathfrak{G}$,

(b) $\mathfrak{G}'_{k-1} = \mathfrak{G}_{k-1}$,

(c) $V_k(\mathfrak{G}') = V_k(\mathfrak{G}) \# T^m$, *where* $m = \dim V_k(\mathfrak{G})$, *and the connected sum is taken along any disc we choose in* $V_k(\mathfrak{G}) - |\mathfrak{A}_k(\mathfrak{G})|$,

(d) $\mathfrak{A}_k(\mathfrak{G}') = \mathfrak{A}_k(\mathfrak{G}) \cup \mathcal{T}^m$, $\mathfrak{A}_{0k}(\mathfrak{G}') = \mathfrak{A}_{0k}(\mathfrak{G}) \cup \mathcal{T}^m$,

(e) $\mathfrak{G}'$ *is seminormal.*

PROOF. T^{m-1} imbeds in $\mathbf{R}^{m+1}$ with trivial normal bundle (in fact it imbeds into $\mathbf{R}^m$ with trivial normal bundle, prove by induction). Hence

$$T^{m-1} \times D^2 \hookrightarrow \operatorname{int}(B^m \times [1,2]) \subset V_k(\mathfrak{G}) \times [1,2],$$

where B^m is a ball in $V_k(\mathfrak{G})$ in the complement of $\mathfrak{A}_k(\mathfrak{G})$. Let C be a thickened arc connecting $B^m \times 1$ to $T^{m-1} \times \partial D^2 = T^m$, so $C \approx B^m \times I$. Let D' be a smaller open disc in D^2, and $A_k = C \cup T^{m-1} \times (D - D')$, and let $\mathcal{B}_k$ be the tico corresponding to $\mathcal{T}^m \times I$ in A_k. Let $\mathfrak{F} = \mathfrak{G} \times [0,2]$, $\mathfrak{F}' = \mathfrak{G} \times [0,1]$, $\mathfrak{F}'' = \mathfrak{G} \times [1,2]$. See Figure 14. By choosing C small we can make sure that $A_k \cap |\mathfrak{A}_k(\mathfrak{F})| = \varnothing$. By Corollary 2.6 we get $\overline{\mathfrak{G}}$ satisfying (a), (b) and

$$V_k(\overline{\mathfrak{G}}) = (V_k(\mathfrak{G}) \# T^m) \amalg T^m \qquad (\amalg \text{ means disjoint union}),$$

$$\mathfrak{A}_k(\overline{\mathfrak{G}}) = (\mathfrak{A}_k(\mathfrak{G}) \cup \mathcal{T}^m) \amalg \mathcal{T}^m.$$

$$\mathfrak{A}_{0k}(\mathfrak{g}) = (\mathfrak{A}_{0k}(\mathfrak{G}) \cup \mathcal{T}^m) \amalg \mathcal{T}^m.$$

By construction $\overline{\mathfrak{G}}$ remains seminormal, so by Proposition 2.1 there exists seminormal $\mathfrak{G}' \in TR$ satisfying the required properties. $\square$
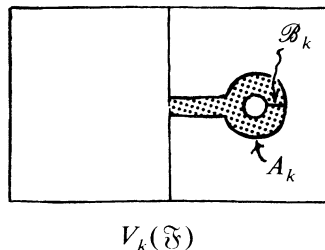

$$V_k(\mathfrak{F})$$

FIGURE 14

3. Construction of spines. The following is a generalization of Proposition 4.2 of [AK_1].

PROPOSITION 3.1. *Let V^m be an m-dimensional compact smooth manifold such that every component of V has nonempty boundary. Let $\mathfrak{A}$ be a closed tico in V such that for any component W of V, $\chi(W - |\mathfrak{A}|) \equiv 0 \pmod 2$, where χ denotes the Euler characteristic. Then there exists a compact codimension zero submanifold A of $\mathrm{int}(V) \times [1, 2)$ and a tico $\mathcal{B}$ in $V \times [0, 1] \cup A$ such that: $\partial\mathcal{B} = \tilde{\mathfrak{A}} \cup \mathcal{C}$ with $|\tilde{\mathfrak{A}}| = |\mathfrak{A}|$ and $\mathcal{C}$ is a closed tico in $\overline{V} = (V \times 1 - \mathrm{int}(\partial_-A)) \cup \partial_+(A)$, and $\mathcal{C}$ is a spine of $\overline{V}$, that is, $\overline{V}/|\mathcal{C}| \approx \mathrm{cone}(\partial\overline{V})$ (notice $\partial\overline{V} = \partial V$). Furthermore, $|\mathcal{B}| \cap V \times [0, 1] = |\mathfrak{A}| \times [0, 1] \cup |\mathcal{B}^{**}|$ where $\mathcal{B}^{**}$ is a separating tico in $V \times [0, 1]$. In fact, $\mathcal{B}^{**} = \{\partial_+D \mid D \in \mathcal{E}\}$ for some codimension zero tim $\mathcal{E}$ in $V \times (0, 1]$ so that $\partial_-\mathcal{E} \subset \mathrm{int}(V) \times 1$. See Figure 15.*
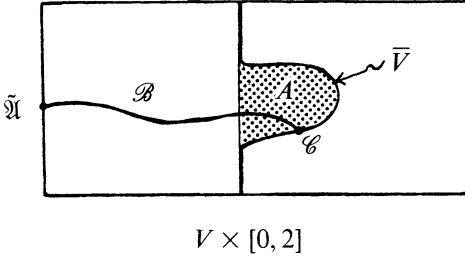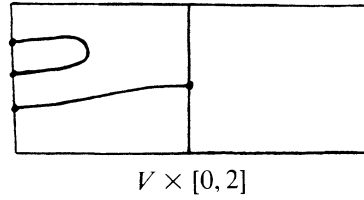

$$V \times [0, 2]$$

FIGURE 15

$$V \times [0, 2]$$

FIGURE 16

PROOF. In case $m = 1$ the proof is trivial. In this case V is a disjoint union of closed intervals and $\mathfrak{A}$ has an odd number of points in each interval. We choose $\mathscr{A}$ and $\mathscr{B}$ in the obvious fashion as in Figure 16.

So we can assume $m \geqslant 2$. Also without loss of generality we will assume that V is connected. Proving the proposition for each component would give the proof for all of V. We claim that we can pick a collection of balls $\mathscr{D} = \{D_i\}_{i=1}^r$ in interior(V) such that (see Figure 17)

(a) $\bigcup_{i=1}^r D_i \cup |\mathfrak{A}|$ is a spine of V.

(b) $\{\partial D_i\}_{i=1}^r \cup \mathfrak{A}$ is a tico in V.

(c) $\bigcup D_i - (\bigcup \partial D_i \cup |\mathfrak{A}|)$ is a union of an even number of open balls $\bigcup_{j=1}^s B_j$, $s = 2p$.
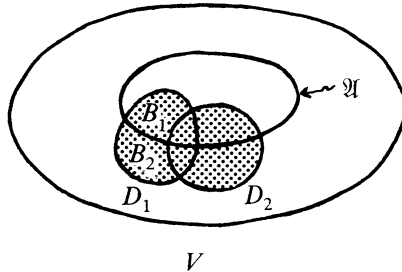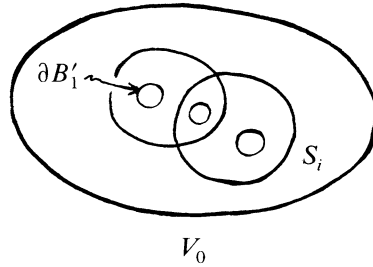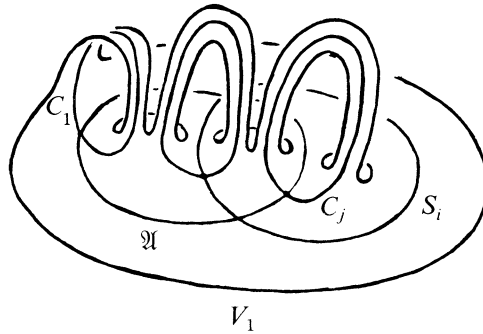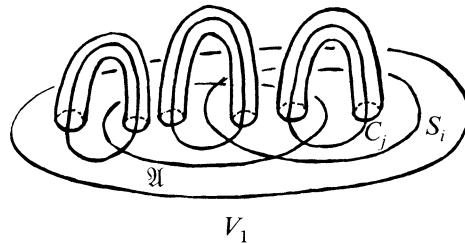

$$V$$

FIGURE 17

FIGURE 18

FIGURE 19

FIGURE 20

Clearly this can be done by covering the complement of an open collar of ∂V by very small round balls. The only thing which is not clear is the evenness claim in (c). We see this as follows. Say s is the number of open balls $\cup D_i - (\cup \partial D_i \cup |\mathfrak{A}|)$; then modulo 2 we have the equations

$$\chi(V) = \chi\left(\bigcup_{i=1}^{r} D_i \cup |\mathfrak{A}|\right) = \chi\left(\bigcup_{i=1}^{r} \partial D_i \cup |\mathfrak{A}|\right) + s = \chi(|\mathfrak{A}|) + s.$$

The last equality can be proven by induction on r. Hence $s = \chi(V - |\mathfrak{A}|)$ so s must be an even integer $2p$.

Call $S_i = \partial D_i$ and let $\mathscr{S} = \{S_i\}_{i=1}^{r} \cup \mathfrak{A}$. $\mathscr{S}$ is a tico in V. Let $B_j' \subset B_j$ be small closed balls; then $V_0 = V - \bigcup_{j=1}^{s}$ interior(B_j') is a manifold with spine $\mathscr{S}$, and $\partial V_0 = \partial V \cup \bigcup_{j=1}^{s} \partial B_j'$, $\partial B_j' \approx S^{m-1}$. See Figure 18. Then we can attach 1-handles to the boundary components of ∂V_0 and get a manifold $V_1 = V_0 \cup$ (1-handles) with $\partial V_1 = \partial V$. V_1 contains the tico $\mathscr{S}$. see Figure 19.

Let $\{C_j\}_{j=1}^{s}$ be the core circles of the 1-handles; then $|\mathscr{S}| \cup \bigcup_{j=1}^{s} C_j$ is a spine of V_1. By construction each C_j intersects a unique element of $\mathscr{S}$. After an isotopy we can see that V_1 is obtained by attaching 1-handles to the interior of V (more correctly surgering S^0's in V). That is, there are closed discs $\{B_l''\}_{l=1}^{2s}$ in $\bigcup_{j=1}^{s} B_j \subset V$ such that $V_1 = (V - \mathrm{int}\bigcup_{l=1}^{2s} B_l'')/\partial B_{2k-1}'' \sim \partial B_{2k}''$ for all $k = 1, 2, \ldots, s$. See Figure 20. Imbed disjoint arcs $(I_k, \partial I_k) \hookrightarrow (V \times [1,2), V \times 1)$ and pick tubular neighborhoods $I_k \times B^m$ of each I_k such that $\partial I_k \times B^m = \{B_{2k-1}'', B_{2k}''\} \times 1$. Let $Z_1 = (V \times [0,1]) \cup \bigcup_{k=1}^{s}(I_k \times B^m)$ where the union is taken along $\partial I_k \times B^m$; then $\partial_{+}(Z_1) = V_1$. See Figure 21.

Let $N(C_j)$ be a closed tubular neighborhood of the circle C_j in V_1; so there are diffeomorphisms $\theta_j : S^1 \times B^{m-1} \to N(C_j)$, $j = 1, \ldots, s$. Recall $s = 2p$ is an even integer. We can extend these imbeddings to imbeddings

$$\Theta_l : \left(S^1 \times B^{m-1} \times I, \, S^1 \times B^{m-1} \times \partial I\right) \hookrightarrow \left(V \times [0,2) - \mathrm{int}(Z_1), Z_1\right)$$

with $\Theta_l|_{S^1 \times B^{m-1} \times \partial I} = \theta_{2l-1}\amalg\theta_{2l}$, $l = 1, 2, \ldots, p$. In other words we pair up the $N(C_j)$'s and pairwise identify them by thickened annuli in the complement of Z_1, i.e. we are attaching round handles to Z_1 in $V \times [1,2)$. See Figure 22.
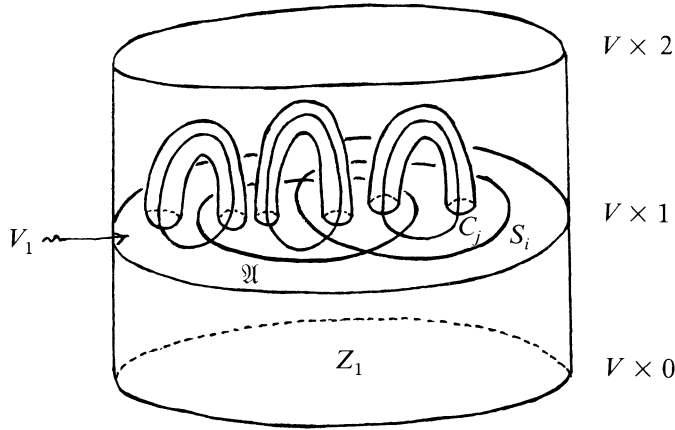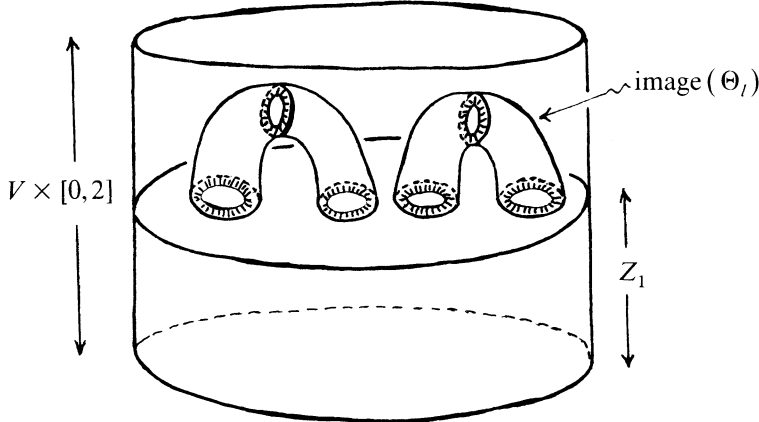

FIGURE 21

FIGURE 22

$$\Theta_l\big(t_0 \times B^{m-1} \times I\big) \longrightarrow \qquad \longleftarrow E_l$$

$$\mathfrak{A}$$

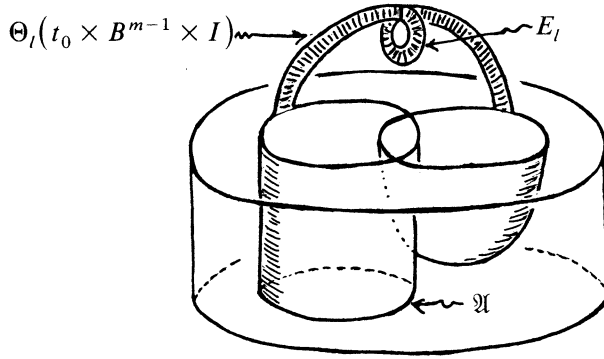FIGURE 23

Let $Z_2 = Z_1 \cup \bigcup_{l=1}^{p} \Theta_l(S^1 \times B^{m-1} \times I)$ where the union identifies $N(C_{2l-1}) \cup N(C_{2l})$ in ∂Z_1 with $\Theta_l(S^1 \times B^{m-1} \times \partial I)$. Call $\partial_+(Z_2) = \overline{V}$. Now define

$$A = \bigcup_{k=1}^{s} (I_k \times B^m) \cup \bigcup_{l=1}^{p} \Theta_l(S^1 \times B^{m-1} \times I).$$

Notice that $\overline{V} = (V \times 1 - \mathrm{int}(\partial_- A)) \cup \partial_+(A)$. Next we will define the tico $\mathscr{B}$.

Recall $\mathscr{S} = \{S_i\}_{i=1}^{r} \cup \mathfrak{A}$; write $\mathfrak{A} = \{S_i\}_{i=r+1}^{q}$. Let $\Lambda_i = \{j \mid C_j \cap S_i \neq \varnothing\}$; then for each $j \in \Lambda_i$, $N(C_j) \cap S_i = \theta_j(t_0 \times B^{m-1}) \approx B^{m-1}$ for some $t_0 \in S^1$.

Define

$$B_i' = \begin{cases} (D_i \times \tfrac{1}{2}) \cup_{\partial} (S^i \times [\tfrac{1}{2}, 1]) & \text{for } i = 1, \dots, r, \\ S_i \times [0, 1] & \text{for } i = r+1, \dots, q. \end{cases}$$

After smoothing the corners $\{B_i'\}$ gives a tico in $V \times [0, 1]$. Construct a 1-complex such that vertices are integers $i = 1, \dots, q$ and connect vertices a and b with an edge if $2l - 1 \in \Lambda_a$ and $2l \in \Lambda_b$ for some $l = 1, \dots, p$. After reordering we can assume that there are $0 = i_0 \leqslant i_1 \leqslant \cdots \leqslant i_\alpha \leqslant q$ such that the vertices of any connected component of this 1-complex lies in an interval of the form $\alpha_n = (i_n, i_{n+1}]$, $n = 0, 1, \dots, \alpha$. Let

$$B_n = \bigcup_{i \in \alpha_n} \left(B_i' \cup \bigcup_{2l \in \Lambda_i} \Theta_l(t_0 \times B^{m-1} \times I) \right), \qquad n = 0, \dots, \alpha.$$

In short, the B_n are a union of B_i''s which are identified by round handles. Let $E_l = \Theta_l(S^1 \times B^{m-1} \times \tfrac{1}{2})$, $l = 1, \dots, p$. Then define $\mathscr{B} = \{B_n\} \cup \{E_l\}$. Then A and $\mathscr{B}$ have the required properties; in particular, $|\mathscr{B}| \cap V \times [0,1] = \bigcup_{i \in \alpha_n} B_i' = |\mathfrak{A}| \times [0, 1] \cup |\mathscr{B}^{**}|$ where $\mathscr{B}^{**}$ is a separating tico. See Figure 23.

In particular, we have

$$\overline{V} = \left(V_1 - \bigcup_{l=1}^{p} \mathrm{int}(C_{2l-1}) \cup \mathrm{int}(C_{2l}) \right) \cup_\partial \bigcup_{l=1}^{p} \Theta_l(S^1 \times S^{m-2} \times I),$$

$$\mathscr{C} = \{\tilde{S}_n\}_{n=0}^{\alpha} \cup \left\{ \Theta_l\big(S^1 \times S^{m-2} \times \tfrac{1}{2}\big) \right\}_{l=1}^{p},$$

where

$$\tilde{S}_n = \bigcup_{i \in \alpha_n} \left[\left(S_i - \bigcup_{j \in \Lambda_i} N(C_j) \cap S_i \right) \cup_\partial \bigcup_{2l \in \Lambda_i} \Theta_l \left(t_0 \times S^{m-2} \times I \right) \right]. \quad \square$$

COROLLARY 3.2. *Let V^m be a compact smooth manifold with boundary such that every component of V has nonempty boundary. Let $\mathfrak{A}$ be a closed tico in V. Let $\mathfrak{A}'$ be a tico in V^m such that $\mathfrak{A} \subset \mathfrak{A}'$ and $\mathfrak{A}$ is a spine of $\mathfrak{A}'$ and if U is any component of $V - |\mathfrak{A}'|$ then $\chi(U) \equiv 0 \pmod 2$. Then there exists A and $\mathscr{B}$ satisfying the conclusion of Proposition 3.1. Furthermore, they can be chosen so that $A \cap |\mathfrak{A}'| \times [0,2] = \varnothing$, $|\mathscr{E}| \cap |\mathfrak{A}' - \mathfrak{A}| \times [0,2] = \varnothing$, $\mathscr{E}$ is a spine of $\mathscr{E} \cup (\mathfrak{A}' \times 1)$, and $\mathscr{B} = \mathfrak{A} \times [0,1] \cup \mathscr{B}'$ where $\mathscr{B}' \cap V \times [0,1] = \mathscr{B}^{**} = \partial_+\mathscr{E}$.*

PROOF. We proceed the same as in the proof of Proposition 3.1 except in the beginning we must choose $\mathscr{D} = \{ D_i \}_{i=1}^r$ so that it satisfies the additional hypothesis $|\mathscr{D}| \subset V - |\mathfrak{A}'|$. Let T be an open regular neighborhood of $|\mathfrak{A}' - \mathfrak{A}|$ in V and let T' be an open collar of ∂V. We can cover $V - T \cup T'$ by collection of small round closed discs $\mathscr{D} = \{ D_i \}_{i=1}^r$ whose boundaries are in general position so that $|\mathscr{D}| \cup |\mathfrak{A}|$ is a spine of V and (a) and (b) of properties $\mathscr{D}$ are satisfied. See Figure 24.

The Euler characteristic condition implies, after a similar argument that $U - \bigcup \partial D_i$ is a union of an even number of balls for each U, and since U meets the boundary, we can

(i) Choose 1-handles $\{ I_k \times B^m \}$ connecting components of $U - \bigcup \partial D_i$, for each U.
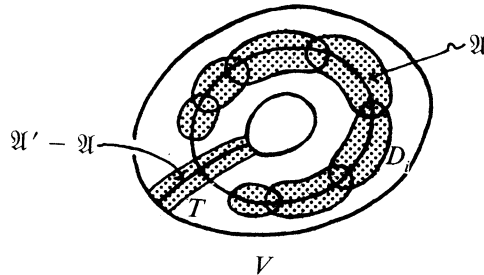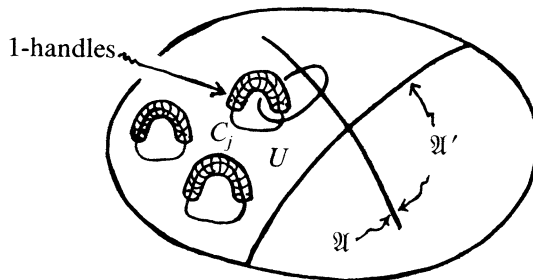

FIGURE 24

FIGURE 25

(ii) Pair up the circles $\{C_j\}$ in each $U \cup$ (1-handles). Then attach the round handles $\{\Theta_l\}$ to these pair of circles, for each U.

Hence we can choose (1-handles) $\cup$ (round handles) $= A$ to miss $\mathfrak{A}' \times [0, 2]$. See Figure 25. By construction $|\mathfrak{A}' \times 1| \cap |\mathscr{C}| = |\mathfrak{A}' \times 1| \cap |\mathfrak{A} \times 1|$, and since $\mathfrak{A}$ is a spine of $\mathfrak{A}'$, then $\mathscr{C}$ is a spine of $\mathscr{C} \cup (\mathfrak{A}' \times 1)$. $\square$

PROPOSITION 3.3. *Let $\mathfrak{F} \in TR$ be seminormal with $V_0(\mathfrak{F}) = $ a point. Suppose that for some $k \geqslant 1$ and all $0 < i < k$ we have diffeomorphisms*

$$h_i : \partial V_i(\mathfrak{F}) \times (0, 1] \to V_i(\mathfrak{F}) - V_{0i}(\mathfrak{F})$$

so that

(a) $h_i(\partial V_{ji}(\mathfrak{F}) \times (0, 1]) = V_{ji}(\mathfrak{F}) - V_{0i}(\mathfrak{F})$ *for all* $0 \leqslant j < i < k$.

(b) $p_{ji} h_i(x, t) = h_j(p_{ji}(x), t)$ *for all* $(x, t) \in \partial V_{ji}(\mathfrak{F}) \times (0, 1]$, $0 < j < i < k$, *where* $p_{ji} = p_{ji}(\mathfrak{F})$.

Then there is a seminormal $\mathfrak{F}' \in TR$ and a diffeomorphism $h_k : \partial V_k(\mathfrak{F}') \times (0, 1] \to V_k(\mathfrak{F}') - V_{0k}(\mathfrak{F}')$ so that

(a') $h_k(\partial V_{jk}(\mathfrak{F}') \times (0, 1]) = V_{jk}(\mathfrak{F}') - V_{0k}(\mathfrak{F}')$ *for all* $0 \leqslant j < k$,

(b') $p'_{jk} h_k(x, t) = h_j(p'_{jk}(x), t)$ *for all* $(x, t) \in \partial V_{jk}(\mathfrak{F}') \times [0, 1]$ *where* $p'_{jk} = p_{jk}(\mathfrak{F}')$,

(c') $\mathfrak{F}'_{k-1} = \mathfrak{F}_{k-1}$,

(d') $\partial \mathfrak{F}' = \partial \mathfrak{F}$.

PROOF. Let $V_i = V_i(\mathfrak{F})$, $V_{ji} = V_{ji}(\mathfrak{F})$, $\mathfrak{A}_i = \mathfrak{A}_i(\mathfrak{F})$, $\mathfrak{A}_{ji} = \mathfrak{A}_{ji}(\mathfrak{F})$, and $p_{ji} = p_{ji}(\mathfrak{F})$. By (a) we know that V_i has no closed components for all $0 < i < k$. So by Proposition 2.1 we may as well assume that V_k has no closed components. If $\dim V_k > 1$ we may also assume that each connected component U of $V_k - |\mathfrak{A}_k|$ has even Euler characteristic. If not, we may use Proposition 2.7 to take the connected sum of U with a torus, thus changing the Euler characteristic of U by one.

Define a map $r : |\mathfrak{A}_k| - V_{0k} \to (0, 1]$ as follows. Suppose $x \in V_{jk} - V_{0k}$. Then we let $r(x) = \pi_j h_j^{-1} p_{jk}(x)$ where $\pi_j : V_j \times (0, 1] \to (0, 1]$ is a projection. By (b) this is well defined since if $x \in V_{jk} \cap V_{ik} - V_{0k}$ with $0 < j < i < k$, then

$$\pi_j h_j^{-1} p_{jk}(x) = \pi_j h_j^{-1} p_{ji} p_{ik}(x) = \pi_i h_i^{-1} p_{ik}(x).$$

Since $\mathfrak{F}$ has type U we know that for each stratum S of $V_{ik} - \bigcup_{j < i} V_{jk}$, $p_{ik}|$: $S \to V_i - |\mathfrak{A}_i|$ is a proper submersion, hence $r| : S \to (0, 1]$ is a proper submersion. Hence we may find a compact neighborhood U of $\bigcup_{i=1}^{k-1} V_{ik}$ in V_k and a proper submersion $r' : U - V_{0k} \to (0, 1]$ so that r is the restriction of r' to $|\mathfrak{A}_k| - V_{0k}$. What we will do now is to modify V_k away from U until r' extends to $V_k - V_{0k}$, then use the submersion r' to get our promised h_k.

Notice $r : |\mathfrak{A}_k| - V_{0k} \to (0, 1]$ is a proper submersion, hence a bundle map, so there is a homeomorphism $g : |\partial \mathfrak{A}_k| \times (0, 1] \to |\mathfrak{A}_k| - V_{0k}$ so that $rg(x, t) = t$, so $\mathfrak{A}_{0k}$ is a spine of $\mathfrak{A}_k$.

First let us consider the case where $\dim V_k = 1$. Then $V_{ik} = \varnothing$ for $0 < i < k$ (since $r| : V_{ik} - V_{0k}$ submerses to $(0, 1]$ and $\dim V_{ik} = 0$, hence $V_{ik} - V_{0k} = \varnothing$). Also by seminormality we know that each component of V_k contains at most one point of V_{0k}. If some component of V_k contains no points of V_{0k}, we may use

Proposition 2.7 to connected sum it with a circle with a point tico. Hence we may as well assume that all components of V_k contain exactly one point of V_{0k}. It is now easy to define h_k. Let it be any diffeomorphism $h_k : \partial V_k \times (0,1] \to V_k - V_{0k}$; conditions (a′) and (b′) are vacuous.

Now suppose $\dim V_k > 1$. By Proposition 3.1 and Corollary 3.2 there is a compact codimension zero submanifold A_k of $V_k \times [1,2)$ and a tico $\mathscr{B}_k$ in $V_k \times [0,1] \cup A_k$ such that if $V_k' = (V_k \times 1 - \mathrm{int}(\partial_- A_k)) \cup \partial_+(A_k)$ then

(1) $A_k \cap |\mathfrak{A}_k| \times [0,2] = \varnothing$,

(2) $|\mathscr{B}_k| \cap V_k'$ is a spine of V_k',

(3) $\mathscr{B}_k = \mathfrak{A}_{0k} \times [0,1] \cup \mathscr{B}_k'$ and $\mathscr{B}_k' \cap V_k \times [0,1] = \partial_+ \mathscr{D}_k$ where $\mathscr{D}_k$ is a codimension zero tim.

(4) $|\mathscr{D}_k| \cap |\mathfrak{A}_{ik}| \times [0,1] = \varnothing$ for all $0 < i < k$,

(5) $|\mathscr{D}_k| \cap V_k \times 0 = \varnothing$ and $|\mathscr{D}_k| \cap \partial V_k \times [0,1] = \varnothing$, i.e. $|\partial_- \mathscr{D}_k| \subset \mathrm{int}(V_k) \times 1$.

(6) $\mathscr{B} \cap V_k'$ is a spine of $\mathfrak{A}_k \times 1$, in fact $|\mathscr{B}_k \cap V_k'| \cap (\bigcup_{j=1}^{k-1} V_{jk} \times 1) = (V_{0k} \cap \bigcup_{j=1}^{k-1} V_{jk}) \times 1$.

So by Corollary 2.6 there exists $\mathfrak{F}' \in TR$ (which is seminormal by construction) satisfying (c′) and (d′). Also $V_k(\mathfrak{F}') = V_k'$, $\mathfrak{A}_{jk}(\mathfrak{F}') = \mathfrak{A}_{jk}(\mathfrak{F}) \times 1$ for $0 < j < k$ and $|\mathfrak{A}_{0k}(\mathfrak{F}')| = |\mathscr{B}_k| \cap V_k'$. Now since $|\mathfrak{A}_{0k}(\mathfrak{F}')|$ is a spine of V_k' we may extend r to be the proper submersion $r'' : V_k' - |\mathfrak{A}_{0k}(\mathfrak{F}')| \to (0,1]$ so that $r'| = r$. Let V_{jk}' denote $|\mathfrak{A}_{jk}(\mathfrak{F}')|$ and let p_{jk}' denote $p_{jk}(\mathfrak{F}')$. Notice $V_{jk}' = V_{jk} \times 1$ for $j > 0$.

We are now ready to start constructing h_k. We will construct h_k by constructing a vector field ω on $V_k' - V_{0k}''$ so that $r_*''(\omega) = 1$; ω is tangent to the sheets of $\mathfrak{A}_k$ and $p_{jk*}'(\omega(x))$ is $\partial h_j(y,t)/\partial t$ where y is such that $p_{jk}'(x) = h_j(y,t_0)$ for some t_0 and $x \in V_{jk}'$. We then obtain h_k by integrating this vector field. It is easy to construct such a ω on $\bigcup_{j=1}^{k} V_{jk}' - V_{0k}'$, using the fact that p_{jk}' submerses all strata to some target stratum (cf. [**AK₄**, Proposition 2.3]). Just construct ω stratum by stratum, starting with the lowest dimensional strata. Now extend ω to all of $V_k' - V_{0k}'$ and normalize so that $r_*'(\omega)$ is still 1.   □

Repeated applications of Proposition 3.3 give the proof of Theorem 1.2 since the hypotheses (a) and (b) of Proposition 3.3 imply that $|\mathfrak{F}_{k-1}| \approx \mathrm{cone}|\partial \mathfrak{F}_{k-1}|$ and the conclusions imply $|\mathfrak{F}_k'| \approx \mathrm{cone}|\partial \mathfrak{F}_k'|$.

COROLLARY 3.4. *Theorem* 1.2 *holds.*

4. Obstructions. Let $\mathscr{S}$ denote the set of isomorphism classes of smooth conelike stratified spaces such that each connected stratum has a trivial normal bundle (these are called TCSS spaces in [**AK₈**]). Let $\mathscr{S}_{PL}$ denote the set of P.L. homeomorphism classes of elements of $\mathscr{S}$; this means that in $\mathscr{S}_{PL}$ a stratified space is equivalent to a smooth subdivision of itself. Recall that TR is the set (of resolution tower isomorphism classes) of compact resolution towers of type RS. Let TR_k denote the elements of TR of dimension $\leq k$. Define η_*^R, η_*^S, η_*^T to be the cobordism groups of TR_*, $|TR_*|$, $|TR_*|_{PL}$ corresponding to the following equivalence relations (1), (2), (3), respectively:

(1) $\mathfrak{F}_1 \sim \mathfrak{F}_2$ if there is $\mathfrak{F} \in TR$ with $\partial \mathfrak{F} = \mathfrak{F}_1 \amalg \mathfrak{F}_2$.

(2) $X \sim Y$ if there are $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F} \in TR$ such that $\partial \mathfrak{F} = \mathfrak{F}_1 \amalg \mathfrak{F}_2$ and $X = |\mathfrak{F}_1|$, $Y = |\mathfrak{F}_2|$.
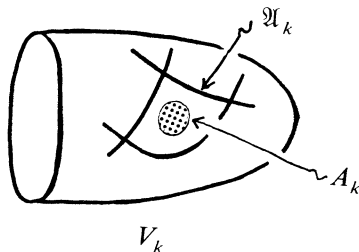
FIGURE 26

(3) $X \sim Y$ if there are $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F} \in TR$ such that $\partial \mathfrak{F} = \mathfrak{F}_1 \amalg \mathfrak{F}_2$ and $X' = |\mathfrak{F}_1|$, $Y' = |\mathfrak{F}_2|$ where X', Y' are some smooth subdivisions of X, Y.

There are obvious surjections $\eta_*^R \to \eta_*^S \to \eta_*^T$. We will denote the cobordism classes by square brackets such as $[\mathfrak{F}]$, $[X]$; it will be clear from the context as to which cobordism group we are referring. It is very likely that if $[\mathfrak{F}] = 0$ in η_*^R then $\mathfrak{F} = \partial \mathfrak{F}'$ for some seminormal $\mathfrak{F}' \in TR$, but we are presently unable to prove this. The following proposition gives strong evidence for this.

PROPOSITION 4.1. *Let* $\mathfrak{F} \in TR_k$ *with* $\mathfrak{F} = \partial \mathfrak{G}$ *for some* $\mathfrak{G} \in TR$. *Then there exists* $\mathfrak{F}' \in TR_k$ *such that* $\mathfrak{F}'$ *and* $\mathfrak{F} \amalg \mathfrak{F}'$ *bound seminormal resolution towers in* TR.

PROOF. Let $\mathfrak{G} = \{V_i, \mathfrak{A}_i, p_i\}_{i=0}^n$. Pick the largest k so that $\mathfrak{G}_{k-1}$ is seminormal. If $k > n$ we are done. We will prove this by induction on $n - k$. Let $A_k \subset \text{int}(V_k) - |\mathfrak{A}_k|$ be a smooth compact codimension zero submanifold so that every component of $V_k - |\mathfrak{A}_k|$ meets $\partial V_k \cup A_k$. See Figure 26.

By Lemma 2.2 there is a compatible family $\{A_i\}_{i=0}^n$ in $\mathfrak{G}$ with $A_i = \varnothing$ for all $i < k$. This gives $\mathfrak{G}''$ and $\mathfrak{G}''' \in TR$ where

$$\mathfrak{G}'' = \{A_i, \mathfrak{A}_i \cap A_i, p_i|\}_{i=0}^n \quad \text{and}$$

$$\mathfrak{G}''' = \{V_i - \text{int}(A_i), \mathfrak{A}_i \cap (V_i - \text{int } A_i), p_i|\}_{i=0}^n.$$

Notice $\partial \mathfrak{G}''' = \mathfrak{F} \amalg \partial \mathfrak{G}''$. Also $_k \mathfrak{G}'''$ and $\mathfrak{G}''_k$ are seminormal. By induction there are $\mathfrak{F}^*$, $\mathfrak{F}^{**} \in TR$ such that $\mathfrak{F}^*$, $\mathfrak{F}^{**}$, $\mathfrak{F} \amalg \mathfrak{G}'' \amalg \mathfrak{F}^*$, $\partial \mathfrak{G}'' \amalg \mathfrak{F}^{**}$ all bound seminormal resolution towers. Then $\mathfrak{F}' = \partial \mathfrak{G}'' \amalg \mathfrak{F}^{**} \amalg \mathfrak{F}^*$ has the required property. $\square$

Hence if we show that bounding seminormal resolution tower is implied by the vanishing of certain additive invariants (characteristic numbers), then Proposition 4.1 would give that if $\mathfrak{F}$ bounds, then it bounds a seminormal resolution tower. This holds for resolution towers of dimension ≤ 2 [$\mathbf{AK_5}$], so we conjecture

CONJECTURE ($*$). If $\mathfrak{F} \in TR$ bounds, then it bounds a seminormal resolution tower.

The positive answer to this conjecture would give a nicer obstruction theory than what we will describe in this paper. To give the reader some idea here we give the values of some of these cobordism groups (see [$\mathbf{AK_5}$]) in Table 1.

TABLE 1

	$i = 0$	$i = 1$	$i = 2$
η_i^T	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^5$
η_i^S	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^{15}$
η_i^R	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^{65}$

Let $\mathscr{S}_n$ denote the elements of $\mathscr{S}$ of dimension $\leqslant n$. Let $Y \in \mathscr{S}_n$ and let Z be an s-dimensional substratified set in Y. A TR-structure on $Y - Z$ is an $\mathfrak{F} \in TR$ such that $|\mathfrak{F}| = Y - Z$; we further demand that $|\mathfrak{F}|$ restricts to a product structure on a tubular neighborhood of each connected stratum of $Y - Z$ and on the boundary of the tubular neighborhood of each s-dimensional connected stratum of Z. Now fix a TR-structure on $Y - Z$. Let σ be an open connected s-dimensional stratum of Z and let $Z' = Z - \sigma$. Then the boundary of the tubular neighborhood of σ is in the form $|\mathfrak{F}| \times \sigma$ for some $\mathfrak{F} \in TR_{n-s-1}$. Then we have

THEOREM 4.2. *The TR-structure of $Y - Z$ extends to $Y - Z'$ if and only if there is a seminormal $\mathfrak{G} \in TR$ such that $\mathfrak{F} = \partial \mathfrak{G}$.*

PROOF. If the structure extends across we would have a TR-structure on the tubular neighborhood cone$|\mathfrak{F}| \times \sigma$ of σ restricting to the old structure $|\mathfrak{F}| \times \sigma$ on the boundary. Hence there exists $\mathfrak{G}$ with $\partial \mathfrak{G} = \mathfrak{F}$ and $|\mathfrak{G}| = $ cone$|\mathfrak{F}|$ so that $\mathfrak{G} \times \sigma$ is the extension. In particular, $\mathfrak{G}$ is seminormal. Conversely if $\mathfrak{F} = \partial \mathfrak{G}$ for some seminormal $\mathfrak{G}$, by Theorem 1.2 there is $\mathfrak{G}'$ with $\partial \mathfrak{G}' = \partial \mathfrak{G}$ and $|\mathfrak{G}'| = $ cone$|\partial \mathfrak{G}| = $ cone$|\mathfrak{F}|$. Hence $\mathfrak{G}' \times \sigma$ gives the required extension. $\square$

Now triangulate Y, let X and Σ be the triangulated copies of Y and Z, respectively. Recall $\dim(\Sigma) = s$. In particular, the boundary of the tubular neighborhood of each open s-simplex of Σ is in the form $|\mathfrak{F}| \times \mathbf{R}^s$ for some $\mathfrak{F} \in TR_{n-s-1}$. That is, the link of each s-simplex has a TR-structure $|\mathfrak{F}|$. Let $\xi(\Sigma)$ be the formal sum of all s-simplexes of Σ with these coefficients in TR_{n-s-1}. In other words, if $\{\sigma_i\}_{i=0}^k$ is the set of all s-simplexes, $\xi(\Sigma) = \sum_{i=0}^k \sigma_i \mathfrak{F}_i$ where $|\mathfrak{F}_i|$ is the link of σ_i. Let $\Sigma' = \Sigma - \bigcup_{i=0}^k \text{int}(\sigma_i)$; we call $\xi(\Sigma)$ the total obstruction to extending the structure to $X - \Sigma'$. Theorem 4.2 say that the TR-structure on $X - \Sigma$ extends to $X - \Sigma'$ if and only if each $\mathfrak{F}_i$ bounds a seminormal resolution tower. If we allow changing the TR-structure on $X - \Sigma$ before extending, we get weaker obstructions to extension as follows. First we need to define the connected sum operation. If $\mathfrak{F}, \mathfrak{F}' \in TR$ with $\dim(\mathfrak{F}) = \dim(\mathfrak{F}') = m$, we can form a new resolution tower $\mathfrak{F} \# \mathfrak{F}'$ by

$$V_i(\mathfrak{F} \# \mathfrak{F}') = \begin{cases} V_i(\mathfrak{F}) \amalg V_i(\mathfrak{F}') & \text{for } i < k, \\ V_m(\mathfrak{F}) \# V_m(\mathfrak{F}') & \text{for } i = m, \end{cases}$$

where the connected sum is taken along discs lying in the interiors of $V_m(\mathfrak{F}) - |\mathfrak{A}_m(\mathfrak{F})|$ and $V_m(\mathfrak{F}') - |\mathfrak{A}_m(\mathfrak{F}')|$, and $p_i(\mathfrak{F}\#\mathfrak{F}')$ defined by restriction. Now let (X, Σ) be as above and let Δ be any $(s + 1)$-simplex of X with $\partial\Delta \subset \Sigma$. Let $\partial\Delta = \sum_{i=0}^{s+1}\sigma_i$, and let the dimension of the link of Δ be $r - 1$ (i.e. the local dimension of X near Δ is $s + 1 + r$). Then the following proposition says that we have some freedom of changing the total obstruction near $\partial\Delta$.

PROPOSITION 4.3. *Given a TR-structure on* $X - \Sigma$ *with the obstruction* $\xi(\Sigma) = \sum_{i=0}^{k}\sigma_i\mathfrak{F}_i$ *and a* $\mathfrak{G} \in TR_r$ *with* $|\mathfrak{G}| \underset{PL}{\approx} S^r$, *then after subdividing* $X - \Sigma$ *we can change the TR-structure near* Δ *such that the new TR-structure on* $X - \Sigma$ *has the obstruction* $\xi'(\Sigma) = \sum_{i=0}^{s+1}\sigma_i(\mathfrak{F}_i\#\mathfrak{G}) + \sum_{i>s+1}^{k}\sigma_i\mathfrak{F}_i$.

PROOF. Let $|\mathfrak{F}|$ be the link of Δ. We have $\dim|\mathfrak{F}| = r - 1$. Recall $|\mathfrak{F}_i|$ are the links of σ_i, $i = 0, \ldots, k$. Let U denote the tubular neighborhood of $\text{int}(\Delta)$. U has a TR-structure by assumption so there is $\mathfrak{F}' \in TR_r$ with $\partial\mathfrak{F}' = \mathfrak{F}$ and $U = |\mathfrak{F}'| \times \text{int}(\Delta)$. Let $\mathfrak{F}'' = \mathfrak{F}'\#\mathfrak{G}$, i.e.

$$V_i(\mathfrak{F}'') = \begin{cases} V_i(\mathfrak{F}') \amalg V_i(\mathfrak{G}) & \text{for } i < r, \\ V_r(\mathfrak{F}')\#V_r(\mathfrak{G}) & \text{for } i = r. \end{cases}$$

Notice $\mathfrak{F}'$ is a codimension zero subtower of each $\mathfrak{F}_i$ for $i = 0, \ldots, s + 1$. So changing the structure on U changes the links of each σ_i to $|\mathfrak{F}^k\#\mathfrak{G}|$ for $i = 0, \ldots, s + 1$ leaves the links of σ_i for $s + 1 < i \le k$ unchanged. See Figure 27. $\square$

Now assume that X has a constant local dimension. This means that the dimension of the link of each i-simplex is $n - i - 1$. Also assume that Σ is the s-skeleton of X. Let $\tilde{\xi}(\Sigma)$ be the chain $\sum_{i=0}^{k}\sigma_i[\mathfrak{F}_i]$ in $C_s(X; \eta_{n-s-1}^R)$. If Conjecture $(*)$ holds then vanishing of $\tilde{\xi}(\Sigma) = 0$ would imply that the TR-structure on $X - \Sigma$ extends to $X - \Sigma'$. We claim that $\tilde{\xi}(\Sigma)$ is a cycle, i.e. $\partial_s\tilde{\xi}(\Sigma) = 0$ where $\partial_s: C_s(X; \eta_{n-s-1}^R) \to C_{s-1}(X; \eta_{n-s-1}^R)$ is the boundary map. This follows by looking at the link of each $(s - 1)$-dimensional simplex of Σ; then the part of the link which lies in the complement of Σ gives the cobordism needed to have $\partial_s\tilde{\xi}(\Sigma) = 0$. See Figure 28.
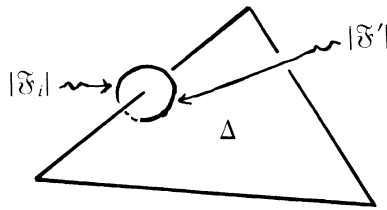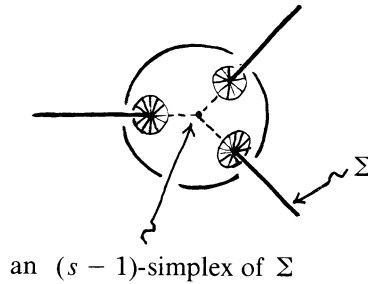

FIGURE 27

an $(s - 1)$-simplex of Σ

FIGURE 28

So $\tilde{\xi}(\Sigma) \in \ker \partial_s$. Let θ_k be the subgroup of η_k^R generated by resolution towers $\mathfrak{G}$ with $|\mathfrak{G}|$ P.L. homeomorphic to S^k. There is a natural map $\rho_{s+1}: C_{s+1}(X; \theta_{n-s-1}^R) \rightarrow C_{s+1}(X; \eta_{n-s-1}^R)$ induced by inclusion, let $\bar{\partial}_{s+1} = \partial_{s+1} \circ \rho_{s+1}$. Proposition 4.3 says that by adjusting the TR-structure on $X - \Sigma$ we can change $\tilde{\xi}(\Sigma)$ by the image of $\bar{\partial}_{s+1}$. Define $\theta_s(X) = \text{kernel}(\partial_s)/\text{image}(\bar{\partial}_{s+1})$. Let $\xi^*(\Sigma) \in \theta_s(X)$ be the image of $\tilde{\xi}(\Sigma)$; then we can summarize the preceding discussion as follows.

COROLLARY 4.4. *If Conjecture* (∗) *holds, then* (a) $\tilde{\xi}(\Sigma) = 0$ *if and only if the* TR-*structure on* $X - \Sigma$ *extends to* $X - \Sigma'$.

(b) *If* $\xi^*(\Sigma) = 0$, *then after possibly changing the structure near the* $(s + 1)$-*skeleton and subdividing* $X - \Sigma$ *the structure on* $X - \Sigma$ *extends to* $X - \Sigma'$.

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